NONLINEAR ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS ON UNBOUNDED DOMAINS

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Solutions are shown to exist for a variety of differential equations. Both ordinary and partial differential equations are considered, with specified initial conditions, boundary conditions, or simultaneous initial and boundary conditions. A key feature of the these problems is a condition at infinity; it is demanded that solutions decay towards zero as the temporal variable becomes arbitrarily large. This feature removes from the problem a certain compactness property, which precludes the use of traditional methods which employ the Leray–Schauder topological degree.

This difficulty is overcome by use of a much newer theory of topological degree, developed by Fitzpatrick, Pejsachowicz, and Rabier in 1992, and later developed further by Pejsachowicz and Rabier in 1998. This degree theory requires several properties in lieu of compactness. It is shown that these properties are available in a wide range of problems, and that there is a practical way to verify this fact in specific cases. Specific examples are given.

Keywords: boundary value problem, initial value problem, infinite interval problem, inhomogeneous problem, nonlinear problem, quasilinear problem, nonautonomous problem, Cauchy problem, Sobolev space, Fredholm operator, proper operator, topological degree, Nemytskii operator, exponential dichotomy, parabolic operator, parabolic evolution equation, holomorphic semigroup, maximal regularity.

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PREFACE

This work is dedicated with love to my wife

CARLA MORRIS

AND to my parents

MARTIN AND CHRYSTELL MORRIS.

This work is presented with gratitude to

CHRIS LENNARD

JUAN MANFREDI

BRYCE MCLEOD

AND VICTOR MIZEL

AND most especially to

PATRICK RABIER.

In the fondest memory of Tom Metzger

1.0 INTRODUCTION

1.1 THE BOUNDARY VALUE PROBLEM ON THE HALF LINE

We will prove that there exist solutions $u \in C^1\big([0, \infty), \mathbb{R}^d\big)$ to certain boundary value problems of the following form:

$$\begin{cases} \dot{u}(t) + F\big(t, u(t)\big) = f(t), & \forall t \geq 0, \\ Pu(0) = \xi, \\ \lim\limits_{t\to\infty} u(t) = 0. \end{cases} \tag{1.1.1}$$

In (1.1.1), $F\colon [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ and $f\colon [0, \infty) \to \mathbb{R}^d$ are given functions, $P\colon \mathbb{R}^d \to \mathbb{R}^d$ is a given linear projection, and $\xi \in \mathbb{R}^d$ is given.

Remark 1.1.1. If the choice $P = I$ is made, then (1.1.1) is the Cauchy (initial value) problem. If $P = 0$, then (1.1.1) is just an ordinary differential equation with no initial condition. Even in this case, the condition as $t \to \infty$ is stronger than mere global existence of a solution, even one that is bounded. A key use of the projection P is in applications to higher order problems. For example, suppose that the dimension $d = 2k$ is even, and that we represent points $x \in \mathbb{R}^{2k}$ as column vectors $x = [x_1,\, x_2]^T$, where $x_i \in \mathbb{R}^k$, $i = 1, 2$. Take F to be of the form

$$F(t,\, x_1,\, x_2) = \begin{bmatrix} -x_2 \\ \\ G(t,\, x_1,\, x_2) \end{bmatrix}, \tag{1.1.2}$$

and f to be of the form

$$f(t) = \begin{bmatrix} 0 \\ \\ g(t) \end{bmatrix}. \tag{1.1.3}$$

Then the first equation in (1.1.1) is equivalent to the second-order equation

$$\ddot{v}(t) + G\big(t,\, v(t),\, \dot{v}(t)\big) = g(t), \tag{1.1.4}$$

where $u(t) = [v(t),\, \dot{v}(t)]^T$. Moreover, the various choices

$$P \begin{bmatrix} x_1 \\ \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \\ x_2 \end{bmatrix},\ \begin{bmatrix} x_1 \\ \\ 0 \end{bmatrix},\ \begin{bmatrix} 0 \\ \\ x_2 \end{bmatrix},\ \text{or}\ \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix} \tag{1.1.5}$$

correspond respectively to initial conditions

$$v(0) = \xi; \quad \dot{v}(0) = \eta \quad \text{(initial value problem)}, \tag{1.1.6}$$

$$v(0) = \xi, \quad \text{(Dirichlet problem)}, \tag{1.1.7}$$

$$\dot{v}(0) = \eta, \quad \text{(Neumann problem), or} \tag{1.1.8}$$

$$\text{(No initial conditions)}. \tag{1.1.9}$$

Of course, other initial conditions are possible. For example, if $P[x_1,\, x_2]^T = [\alpha x_1 + \alpha x_2,\, 0]$, then the condition $Px = [\xi,\, 0]^T$ would correspond to $\alpha v(0) + \beta \dot{v}(0) = \xi$. That $u(t) \to 0$ as $t \to \infty$ corresponds to the requirement that both $v(t)$ and $\dot{v}(t)$ tend to zero as $t \to \infty$. $\diamondsuit$

Remark 1.1.2. Even with the conditions that we will place on F (see the three boxed conditions on page 16), it would not be less general to replace the right side $f(t)$ in (1.1.1) by 0, since $f(t)$ can be absorbed into $F\big(t,\, u(t)\big)$. However, the spirit of the main existence result (Theorem 2.4.1 on page 83) is that F is such that the problem (1.1.1) has some special feature when $f(t) = 0$ that need not be shared for other choices of f. Even if these features are lost as 0 is deformed into f, the existence of solutions is not lost. For a more detailed explanation of exactly which features we might be talking about, see Item 5 of Theorem 2.4.1 on page 83, and also Remarks 2.4.2 and 2.4.3 that follow the proof of the theorem. $\Diamond$

Remark 1.1.3. Quite recently, problem (1.1.1) was studied by Rabier and Stuart [RSb, RSa], but in the setting of Sobolev spaces and with stronger conditions on the asymptotic behavior of F. Also, because we work in $C^1_{\{0\}}$, our results can be easier to apply because *a priori* bounds are generally easier to obtain with the $C^1_{\{0\}}$ norm than with the $W^{1,p}$ norm. One reason for this is that when $u \in C^1_{\{0\}}$, the derivative $\dot{u}$ is well-defined pointwise. We can then take advantage of the fact that $\big\langle \dot{u}(t),\, \dot{u}(t)\big\rangle = 0$ at an interior maximum of u. See Section 2.5 for examples.

Rabier and Stuart discuss the status of boundary value problems on the half-line in [RSb]. As they point out, problems on the half line have also been investigated by Andres, Gabor, and Górniewicz in [AGG99], which contains numerous references, especially to earlier work. Another good survey of the state of the problem can be found in the recent book [AO01] of Agarwal and O'Regan. This book points out numerous physical problems that lead to boundary value problems on the half-line. The emphasis is on second order scalar equations, although systems are also discussed.

The literature generally gives existence results in spaces that do not capture the desired behavior at infinity. Any condition at infinity then has to be obtained by a direct analysis of particular problems. Some exceptions exist; for example, Agarwal and O'Regan discuss a special case of (1.1.1) in section 1.12 of [AO01]. However they include a t-integrability condition on F that rules out autonomous problems. On the other hand, in [AO02] Agarwal and O'Regan study an autonomous problem

$$\begin{cases} \ddot{v}(t) - 2\sinh v(t) = 0, \qquad \forall t \geq 0, \\ v(0) = \xi, \\ \lim\limits_{t\to\infty} v(t) = 0 \end{cases} \tag{1.1.10}$$

that arises in the theory of colloids, when relating particle stability with the charge on the colloidal particle. Agarwal and O'Regan prove that a solution to this problem (as one of a class of problems) exists. Still, we point out that the solubility of (1.1.10), and also of suitable time-dependent variations is covered by our Theorem 2.4.1. See Section 2.5.3 for details. $\Diamond$

1.2 THE CAUCHY PROBLEM IN THE SEMI-INFINITE CYLINDER

We will prove the existence of solutions $u = u(t, x)$ in $W^{1,p}\big([0, \infty) \times \Omega\big)$ to boundary value problems of the following form:

$$u_t(t, x) - A(t)u(t, x) - G\big(t, u(t, x)\big) = f(t, x), \qquad t \geq 0,\, x \in \Omega; \tag{1.2.1a}$$

$$u(0, x) = g(x), \qquad x \in \Omega; \tag{1.2.1b}$$

$$u(t, x) = 0, \qquad t \geq 0,\, x \in \partial\Omega, \tag{1.2.1c}$$

$$\lim_{t\to\infty} \sup_{x\in\Omega} \big|u(t, x)\big| = 0. \tag{1.2.1d}$$

Here, f and g are given functions (drawn from a space to be determined), $A(t)$ is a differential operator in Ω, and $G = G(t, \xi)$ is some nonlinearity. We will work specifically with the case that A is a second order differential operator, so we seek solutions u such that the second-order distributional derivatives of u in the x-variables are $L^p\big([0, \infty) \times \Omega\big)$ functions. We do not require this of the second-order distributional derivatives of u that involve the t-variable, and so we work in an anisotropic Sobolev space.

This problem was studied quite recently in the linear setting ($G = 0$) by Rabier, first in the autonomous setting $A(t) = A_0$ on the whole line [Rab03], and then in the nonautonomous setting on the half line [Rab04b]. Our approach in the nonlinear setting will be to reformulate problem (1.2.1) as an evolution problem for $\mathbf{u} = \mathbf{u}(t)$, where for each t, $\mathbf{u}(t)$ represents the partial map $u(t, \cdot)$. This will transform (1.2.1) into a new problem of the form

$$\dot{\mathbf{u}} - A^\sharp(\mathbf{u}) - G^\sharp(\mathbf{u}) = \mathbf{f} \tag{1.2.2}$$

$$\mathbf{u}(0) = g. \tag{1.2.3}$$

This problem turns out to share many key features with the problem (1.1.1) of the preceding section. Indeed, we will prove an existence result by adaptation of the methods used in the solution of problem (1.1.1). The main difficulty lies in the fact that $\mathbf{u}$ takes values not in $\mathbb{R}^d$, but instead in an infinite dimensional Banach space.

1.3 NOTATION AND GENERAL BACKGROUND

Before proceeding, we set some notation to be used throughout the sequel. The complex and real fields, the integers, and the natural numbers are denoted by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$, respectively.

All vector spaces are taken to be over the field $\mathbb{R}$, unless otherwise noted. The natural number d will be reserved for the dimension of $\mathbb{R}^d$. On $\mathbb{R}^d$ we will use the norm $|x| = \sum_i |x_i|$. In particular, for $(t, x) \in \mathbb{R} \times \mathbb{R}^d$, we use $|(t, x)| = |t| + |x|$. This choice of norm in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ induces the associated matrix norm in $\mathbb{R}^{d_1 \times d_2}$: $|A| = \sup\{|Ax| \mid |x| = 1\}$. These different uses for $|\cdot|$ should cause no confusion, but we will sometimes write $\|A\|_{\mathrm{op}}$ instead of $|A|$ to emphasize that this is an induced operator norm.

We shall have use for the following function spaces. In this, we have striven to follow the notation used by Evans [Eva98]; see in particular Appendix A of that book. If Ω is an open subset of a normed linear space $(W, \|\cdot\|_W)$ and $(X, \|\cdot\|_X)$ is a normed linear space, then $C(\Omega, X)$ denotes the vector space of all continuous functions from Ω into X. The space of all functions $f\colon \Omega \to X$ that have continuous derivatives through order $k \in \mathbb{N}$ is denoted by $C^k(\Omega, X)$. For emphasis, we will often write $C^0(\Omega, X)$ instead of $C(\Omega, X)$. The intersection of all of the spaces $C^k(\Omega, X)$ is $C^\infty(\Omega, X)$. The subspace of $C^\infty(\Omega, X)$ consisting of all functions in C^∞ that vanish outside of a compact subset of Ω is denoted by $C_0^\infty(\Omega, X)$. Let $\overline{\Omega}$ denote the topological closure of Ω. If $0 \le k < \infty$, we denote by $C^k(\overline{\Omega}, X)$ the space of all $f \in C^k(\Omega, X)$ such that all derivatives of f (through order k) are uniformly continuous on bounded subsets of Ω.

For any $f \in C(\overline{\Omega}, X)$, put

$$\|f\|_\infty := \sup\{\|f(w)\|_X : \mid w \in \Omega\} \in [0, \infty]. \tag{1.3.1}$$

We will make frequent use of the following Banach spaces associated with $\|\cdot\|_\infty$. The subspace of $C(\overline{\Omega}, X)$ for which $\|f\|_\infty$ is finite is denoted by $C_b(\overline{\Omega}, X)$. This space is complete under the norm $\|\cdot\|_\infty$. If $f \in C^1(\overline{\Omega}, X)$, define

$$\|f\|_{\infty,1} := \|f\|_\infty + \|Df\|_\infty \in [0, \infty]. \tag{1.3.2}$$

The space of all $f \in C^1(\overline{\Omega},\, X)$ for which $\|f\|_{\infty,1}$ is finite will be denoted by $C^1_b(\overline{\Omega},\, X)$. This space is a Banach space under the norm $\|\cdot\|_{\infty,1}$.

If Ω is an unbounded domain in $\mathbb{R}^d$, we denote by $C_{\{0\}}(\overline{\Omega},\, X)$ the space of all functions $f \in C_b(\overline{\Omega},\, X)$ such that

$$\lim_{\substack{|w|\to\infty \\ w\in\Omega}} \|f(w)\| = 0. \tag{1.3.3}$$

This is a closed vector subspace of $C_b(\overline{\Omega},\, X)$, and is hence a Banach space. Similarly, we denote by $C^1_{\{0\}}(\overline{\Omega},\, X)$ that subspace of $C^1_b(\overline{\Omega},\, X)$ consisting of all f such that

$$\lim_{\substack{|w|\to\infty \\ w\in\Omega}} \|f(w)\| + \|Df(w)\| = 0. \tag{1.3.4}$$

With all of the above spaces, if the target space X is known from the context, as will most often be the case, it will be dropped from the notation: $f \in C_{\{0\}}(\overline{\Omega})$, for example. We will sometimes drop the domain from the notation as well: $f \in C_{\{0\}}$. Such symbols can also be used as adjectives. To say that f is a C^1 map from Ω into X is synonymous with $f \in C^1(\Omega,\, X)$.

We will also use some function spaces that arise in the study of partial differential equations on $\Omega \subset \mathbb{R}^d$. In particular, if $0 \le \lambda \le 1$, we denote by $C^{0,\lambda}(\overline{\Omega}, X)$ that space of all $f \in C(\overline{\Omega},\, X)$ such that for some $M = M(f) > 0$,

$$\|f(w) - f(w')\|_X \le M\,|w - w'|^\lambda, \quad \forall w, w' \in \Omega. \tag{1.3.5}$$

If one takes $\|f\|_{0,\lambda} = \|f\|_\infty + \mu$, where μ is the infimum over all valid values of M in (1.3.5), then $C^{0,\lambda}(\overline{\Omega}, X)$ is a Banach space known as a Hölder space.

We make much use of the Lebesgue and Sobolev spaces $L^p(\Omega,\, \mathbb{R}^n)$ and $W^{k,p}(\Omega,\, \mathbb{R}^n)$. For definitions and standard properties, see Rudin [Rud87], Adams and Fournier [AF03], and Evans [Eva98]. For the spaces $L^p(\Omega, X)$ and $W^{k,p}(\Omega, X)$ when X is an infinite dimensional Banach space, one uses the theory of the Bochner integral. For this, see Diestel and Uhl [DU77], Dunford and Schwartz [DS88], and Edwards [Edw65]. For a concise treatment of some of the most relevant properties, see Appendix E.5 and Section 5.9 of Evans [Eva98]. Recall that the notation $W_0^{k,p}(\Omega)$ is used to denote the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$. When k and p are mutually large enough with respect to d that functions in $W^{k,p}(\Omega)$ have some smoothness ($k \geq 1$ and $p \geq 1$ work if $d = 1$) then it makes sense (and is correct) to say that functions $W_0^{k,p}(\Omega)$ vanish on the boundary of Ω. When a function takes values in an infinite dimensional Banach space, we will usually use a bold typeface as in $\mathbf{u}\colon [0, \infty) \to L^p(\Omega)$ for emphasis.

The space of all continuous linear operators from a normed linear space $(W, \left\|\cdot\right\|_W)$ into a normed linear space $(X, \left\|\cdot\right\|_X)$ will be denoted by $\mathcal{L}(W, X)$. If $L \in \mathcal{L}(W, X)$, we define the usual operator norm

$$\left\|L\right\|_{\text{op}} := \sup_{\left\|w\right\|_W = 1} \left\|Lw\right\|_X. \tag{1.3.6}$$

If Y is a Banach space, then $L \in \mathcal{L}(W, X)$ is a Banach space when equipped with the above norm. As usual, we write $\mathcal{L}(W)$ instead of (W, W), and we use W^* for the dual space $\mathcal{L}(W, \mathbb{R})$ (or for $\mathcal{L}(W, \mathbb{C})$ if W is a complex Banach space). If a sequence (y_n) in W is such that for some $y \in W$, we have $\phi(y_n) \to \phi(y)$ for all $\phi \in W^*$, then we say that (y_n) converges weakly to y and we write[1] $y_n \xrightharpoonup{w} y$. See Dunford and Schwartz [DS88], Rudin [Rud91], or Kreyszig [Kre89] for more standard terminology and results from linear functional analysis.

For an interval $I \subset \mathbb{R}$, it is often convenient to associate with a map $F : I \times \mathbb{R}^d \to \mathbb{R}^d$ the following operator, said to be the Nemytskii operator associated to F. For any function

[1] The "w" above the arrow stands for "weakly," not for the space W.

$u : I \to \mathbb{R}^d$, define $\widetilde{F}(u) : I \to \mathbb{R}^d$ by

$$\widetilde{F}(u)(t) := F\big(t,\, u(t)\big). \tag{1.3.7}$$

In such generality, this is just notation. However, we shall see that useful properties of $\widetilde{F}$ can be deduced from appropriate properties of F and of the function space in which u is found. We will prove all properties that we use herein. Nemytskii operators were studied by Marcus and Mizel [MM73, MM75] and used more recently by Rabier and Stuart[RSb, RSa] in a different setting.

1.4 THE FREDHOLM AND PROPERNESS PROPERTIES

Definition 1.4.1. Let X and Y be Banach spaces, and let L be a continuous linear operator from X into Y. The operator L is said to be Fredholm, and to have the Fredholm property, if $\ker L \subset X$ is finite dimensional and $\operatorname{rge} L$ is of finite co-dimension in Y. In this case, the Fredholm index k of L is defined to be

$$k = \dim \ker L - \operatorname{codim}_Y \operatorname{rge} L. \tag{1.4.1}$$

Remark 1.4.2. Notice that this definition depends critically on the choice of both X and Y. For example, if Y is a closed subspace of a Banach space W and we view L as a map into W, the Fredholm property changes. Specifically, if W/Y has finite dimension n, the index of L is reduced by exactly n. If W/Y is infinite dimensional, then $L : X \to W$ is not even Fredholm. $\Diamond$

For properties of linear Fredholm operators, see section IV.5 in Kato [Kat95]. To pass from linear operators to nonlinear operators, the key item is that the Fredholm property and index is stable under perturbations of sufficiently small norm. Thus, the set of all linear operators of a particular index is open in $\mathcal{L}(X, Y)$. As a result, any connected set of Fredholm operators is such that all of its members have a common index. This justifies the following definition:

Definition 1.4.3. Let X and Y be Banach spaces, and let Φ be a continuously differentiable map from X into Y. If $D\Phi(u)$ is Fredholm of some index at every point $u \in X$, then the index K of $D\Phi(u)$ is independent of u, and then the operator Φ is said to be Fredholm of index k. ♦

We will make frequent use of the following two properties of the Fredholm index, the first of which is trivial.

Property 1.4.4. *Let X and Y be Banach spaces, and let L be a linear isomorphism from X onto Y. Then L is Fredholm of index zero. Consequently, if Φ is a continuously differentiable map from X into Y such that $DF(u)$ is Fredholm at every point $u \in X$ and such that $DF(u)$ is an isomorphism of X onto Y at some point $u \in X$, then Φ is Fredholm of index zero.*

Property 1.4.5. *Let X and Y be Banach spaces, and let L be a linear Fredholm operator from X into Y. Let K be a compact linear operator[2] from X into Y. Then $L+K$ is Fredholm of the same index.*

We next define what it means for an operator to be proper:

Definition 1.4.6. Let X and Y be complete metric spaces. A continuous map $F\colon X \to Y$

[2]This means that K maps bounded sets onto relatively compact sets. Equivalently, the image under K of each X-bounded sequence has a Y-convergent subsequence.

is said to be *proper* if $F^{-1}(K)$ is compact in X whenever K is compact in Y.

Example 1.4.7. In case $X = Y = \mathbb{R}$ with the standard metric, non-constant polynomials are proper, but the sine and cosine functions are not. The exponential function is proper on $[0, \infty)$, but not on $\mathbb{R}$. All continuous functions are proper on $[0, 1]$.

Properness on each closed, bounded subset of a Banach space will be of particular interest in Section 2.2. The following property describes a standard technique for proving this property.

Property 1.4.8. *Let X and Y be Banach spaces, and let $F\colon X \to Y$ be given. The following are equivalent:*

1. *The restricted map $F\big|_S$ is proper for each closed, bounded subset S of X.*
2. *If (x_n) is a bounded sequence in X such that $\big(F(x_n)\big)$ is convergent in Y, then (x_n) has a convergent subsequence.*

Proof. This result is standard, but for completeness, we sketch the elementary proof. If statement 1 holds, take S to be a closed ball containing the sequence (x_n). Since $\overline{\{F(x_n)\colon n \in \mathbb{N}\}}$ is compact in Y, its pre-image W in S is therefore compact. Since $(x_n) \subset W$, the first implication is proved.

Conversely, let S be a given closed, bounded subset of X. Given a compact subset K of Y, we are to show that $C = F^{-1}(K) \cap S$ is compact in X. If (x_n) is any sequence drawn from C, then $\big(F(x_n)\big) \subset K$ has a convergent subsequence. According to statement 2, the corresponding subsequence of (x_n) then has a further subsequence that converges. Thus, C is compact. $\square$

We will use one more property, which involves both the Fredholm and properness properties. This is known as Yood's criterion. See Deimling [Dei85], Proposition 9.3 for a proof.

Property 1.4.9. *Let L be a continuous linear map from the Banach space X into the Banach space Y. The following are equivalent:*

1. *The restricted map $F\big|_S$ is proper for each closed, bounded subset S of X.*
2. *The range of L is closed in Y, and the kernel of L is finite-dimensional.*

1.5 THE TOPOLOGICAL DEGREE OF FREDHOLM MAPPINGS

For an introduction to the topological degree for continuous maps on finite-dimensional spaces (Brouwer's degree), see Deimling [Dei85] or Lloyd [Llo78]. See also the author's Master's thesis[Mor01].[3] The most widely known extension of the Brouwer degree to infinite-dimensional spaces is the Leray-Schauder degree for compact perturbations of the identity. The books of Deimling [Dei85] and Lloyd [Llo78] also treat the Leray-Schauder degree.

The Leray-Schauder degree is not applicable to an operator that is not a compact perturbation of the identity. For the problems that we study here, the relevant compactness properties are absent. This is because the domains for the differential equations are unbounded. We will use the degree theory of C^1 proper Fredholm maps, developed by Fitzpatrick, Pejsachowicz, and Rabier in [FPR92] for C^2 maps, and later extended to C^1 maps by Pejsachowicz and Rabier in [PR98]. For a survey of other theories of topological degree and how they relate to this theory, see the paper [FPR94] of Fitzpatrick, Pejsachowicz, and

[3]The author exposits the definition of Brouwer's degree in full detail, and also exposits an application of the degree due to Hadeler [Had71].

Rabier. We use only the following portion of this degree theory, which we present here for reference.

Definition 1.5.1. Let X and Y be real Banach spaces. Let Ξ denote the set of all triples $(F,\,\Omega,\,y)$ that satisfy the following three conditions:

1. Ω is an open subset of X.
2. F is a map from X into Y such that:
    a. $F \in C^1(X,\,Y)$.
    b. F is Fredholm of index zero.
    c. $F\big|_{\overline{\Omega}}$ is proper.
3. $y \in Y \setminus F(\partial\Omega)$. ♦

There is a well-defined map $\left|d\right| : \Xi \to \mathbb{N} \cup \{0\}$ that satisfies the following properties. We say that $\left|d\right|(F,\,\Omega,\,y)$ is "the absolute degree of F at y relative to Ω." In each property, the use of the notation $\left|d\right|(F,\,\Omega,\,y)$ involves the implicit assumption that $(F,\,\Omega,\,y) \in \Xi$.

Property 1.5.2 (Normalization). *If $\left|d\right|(F,\,\Omega,\,y) \neq 0$, then there is $x \in \Omega$ such that $F(x) = y$.*

Remark 1.5.3. Note that the solution x to $F(x) = y$ is not asserted to be unique. Also notice that x may or may not exist if $\left|d\right|(F,\,\Omega,\,y) = 0$. ♢

Property 1.5.4 (Homotopy invariance). *Let $h \in C^1\big([0,\,1] \times X,\,Y\big)$ be Fredholm of index 1, such that $h\big|_{[0,\,1]\times\overline{\Omega}}$ is proper. If $y \in Y \setminus h\big([0,\,1] \times \partial\Omega\big)$, then*

$$\left|d\right|\big(h(0,\,\cdot),\,\Omega,\,y\big) = \left|d\right|\big(h(s,\,\cdot),\,\Omega,\,y\big) = \left|d\right|\big(h(1,\,\cdot),\,\Omega,\,y\big) \tag{1.5.1}$$

for all $s \in [0,\,1]$.

Remark 1.5.5. The use of Property 1.5.2 and Property 1.5.4 is how the degree is used as an existence tool. If we can deform F into another function G in a way consistent with Property 1.5.4, then $\left|d\right|\big(F,\,\Omega,\,y\big) = \left|d\right|\big(G,\,\Omega,\,y\big)$. If also $\left|d\right|\big(G,\,\Omega,\,y\big)$ is known to be nonzero, then Property 1.5.2 ensures the existence of a solution $x \in \Omega$ to the equation $F(x) = y$. The remaining two properties are helpful in showing that $\left|d\right|\big(G,\,\Omega,\,y\big) \neq 0$ in the first place. (Recall from Remark 1.5.3 that it is insufficient in general to know merely that $G(x) = y$ is solvable in Ω, although this is necessary.) $\diamondsuit$

Property 1.5.6 (Converse normalization). *If there is exactly one $x \in \Omega$ such that $F(x) = y$, and if $DF(x)$ is a linear isomorphism of X onto Y, then $\left|d\right|(F,\,\Omega,\,y) = 1$.*

Property 1.5.7 (Borsuk's theorem). *If $F(x) = -F(-x)$ for all $x \in X$ (so that F is said to be "odd"), and if B_R is the open ball of radius $R > 0$ centered at the origin of X, then $\left|d\right|(F,\,B_R,\,0)$ is odd. In particular,*

$$\left|d\right|(F,\,B_R,\,0) \neq 0. \tag{1.5.2}$$

2.0 BOUNDARY VALUE PROBLEMS ON THE HALF LINE

We first turn to the boundary value problem discussed in Section 1.1:

$$\begin{cases} \dot{u}(t) + F\bigl(t,\, u(t)\bigr) = f(t), \qquad \forall t \geq 0, \\ Pu(0) = \xi, \\ \lim\limits_{t\to\infty} u(t) = 0. \end{cases} \tag{2.0.1}$$

We will prove an existence theorem by using the topological degree theory that was reviewed in Section 1.5. We begin in Section 2.1 by reformulating (2.0.1) as the problem of finding u in a Banach space X such that $\Phi(u) = y$, for a suitable Banach space Y and operator $\Phi\colon X \to Y$. We also give conditions on F such that Φ will have the smoothness necessary for use of the degree theoretical argument.

Then, in Section 2.2, we find conditions on F such that Φ will have the required properness property. Following the lead of Rabier and Stuart in [RSb, RSa], we prove a properness criterion based on solubility of homogeneous equations $\dot{u}(t) + G\bigl(t,\, u(t)\bigr) = 0$, where the functions G are obtained as suitable limits of sequences of translates $\bigl(F(\cdot + \sigma_n,\, \cdot)\bigr)$.

In Section 2.3 we study the Fredholm property and Fredholm index of Φ, which must be Fredholm of index zero for use of the topological degree argument. These considerations are shown to depend on the structure of solutions to the linearization of (2.0.1) at $u = 0$, which is obtained by replacing $F(t, x)$ by $D_x F(t, 0)x$. Various sufficient conditions are developed; some of these are also shown to be necessary.

At that point, we state and prove the existence Theorem 2.4.1 in Section 2.4. Several remarks are also given which show how the hypotheses can be further simplified in some important special cases.

In Section 2.5 we show how Theorem 2.4.1 can be used to prove existence in specific problems. To warm up, we consider a first order scalar equation in Section 2.5.1. We then consider first-order systems of equations in Section 2.5.2, where we study problem with functions of the form $F(t, x) = g(t)\nabla\phi(x)$. In Section 2.5.3 we prove existence for a class of second-order problems, both with Dirichlet and with Neumann boundary conditions.

2.1 SMOOTHNESS OF THE NEMYTSKII OPERATOR

We will often assume in this and later sections that $F = F(t, x) : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is a function that satisfies each of the following conditions:

$$\begin{gathered} F \text{ is continuous on } [0, \infty) \times \mathbb{R}^d \text{ and} \\ D_x F \text{ exists and is continuous on } [0, \infty) \times \mathbb{R}^d; \end{gathered} \tag{2.1.1a}$$

$$\begin{gathered} F \text{ and } D_x F \text{ are bounded and uniformly continuous} \\ \text{on } [0, \infty) \times K, \text{ for each compact } K \subset \mathbb{R}^d; \end{gathered} \tag{2.1.1b}$$

$$\lim_{t \to \infty} F(t, 0) = 0. \tag{2.1.1c}$$

Remark 2.1.1. Since we can subtract $f_0(t) := F(t,\, 0)$ from each side of the inhomogeneous equation (2.0.1), we could as well assume that $F(t,\, 0) = 0$ instead of (2.1.1c). On the other hand, assumption (2.1.1c) does not cause much us difficulty, and in principle allows slightly more flexibility in applications. $\Diamond$

2.1.1 The Nemytskii operator and its derivative

We associate to each $u \in C^1_{\{0\}}\big([0,\, \infty),\, \mathbb{R}^d\big)$ a function $\widetilde{F}(u)$ via the definition

$$\widetilde{F}(u)(t) := F\big(t,\, u(t)\big). \tag{2.1.2}$$

The map $\widetilde{F}$ that carries u into $\widetilde{F}(u)$ is known as the Nemytskii operator associated with F. The above assumptions provide $\widetilde{F}$ with the desired range and smoothness properties:

Theorem 2.1.2. *Suppose that F satisfies (2.1.1a), (2.1.1b), and (2.1.1c). Then the associated Nemytskii operator $\widetilde{F}$ is a continuously differentiable map from $C^1_{\{0\}}\big([0,\, \infty),\, \mathbb{R}^d\big)$ into $C_{\{0\}}\big([0,\, \infty),\, \mathbb{R}^d\big)$. Moreover, the derivative $D\widetilde{F}(u)$ is given by*

$$\Big(\big[D\widetilde{F}(u)\big]h\Big)(t) = D_xF(t,\, u(t))h(t), \quad \forall h \in C^1_{\{0\}}, \forall t \geq 0. \tag{2.1.3}$$

Briefly,[1]

$$D\widetilde{F}(u)h = \widetilde{D_xF}(u)h, \quad \forall h \in C^1_{\{0\}}. \tag{2.1.4}$$

[1] The left hand side features application of an operator to a vector, while the right hand side represents pointwise (in t) multiplication.

Proof. We first show that $\widetilde{F}$ takes values in $C_{\{0\}}$. Let $u \in C^1_{\{0\}}$ be given. To show that $\widetilde{F}(u)$ is continuous, let $t \geq 0$, and suppose that $t_n \to t$ in $[0, \infty)$. Then, because u is continuous, $\big(t_n, u(t_n)\big) \to \big(t, u(t)\big)$ in $[0, \infty) \times \mathbb{R}^d$. Hence,

$$\widetilde{F}(u)(t_n) = F\big(t_n, u(t_n)\big) \to F\big(t, u(t)\big) = \widetilde{F}(u)(t) \tag{2.1.5}$$

as $n \to \infty$, using the continuity of F at the point $\big(t, u(t)\big)$.

To show that $\widetilde{F}(u)(t) \to 0$ as $t \to \infty$, let $\epsilon > 0$. Since u is bounded, there is a compact set K which contains the range of u in $\mathbb{R}^d$. By the uniform continuity of F on $[0, \infty) \times K$ (see (2.1.1b)), along with the convergence of $u(t)$ to $0 \in K$ as $t \to \infty$, there is some $T > 0$ such that

$$\big|F\big(t, u(t)\big) - F(t, 0)\big| < \epsilon/2, \quad \forall t \geq T. \tag{2.1.6}$$

Also, by increasing T if necessary, assumption (2.1.1c) ensures that

$$\big|F(t, 0)\big| < \epsilon/2, \quad \forall t \geq T. \tag{2.1.7}$$

Together, (2.1.6) and (2.1.7) show that $\big|F\big(t, u(t)\big)\big| < \epsilon$ if $t > T$, which proves $\widetilde{F}(u) \in C_{\{0\}}$.

Next, we show that $\widetilde{F}$ is continuous on $C^1_{\{0\}}$. Let (u_n) be a sequence that converges in $C^1_{\{0\}}$ to some $u \in C^1_{\{0\}}$. Since such a sequence is bounded in $C^1_{\{0\}}$, there exists a compact subset K of $\mathbb{R}^d$ which contains the range of u and also the union of the ranges of all of the functions u_n. Let $\epsilon > 0$. By the uniform continuity of F on $[0, \infty) \times K$ (assumption (2.1.1b)), there is some $\delta > 0$ such that $\big|F(t, x) - F(t, x')\big| < \epsilon$ for all $t \geq 0$ and all $x, x' \in K$ such that $\big|x - x'\big| < \delta$. For this δ, the (uniform) convergence of (u_n) to u gives some $N \in \mathbb{N}$ such that $\big|u_n(t) - u(t)\big| < \delta$ for all $t \geq 0$ and $n \geq N$. Hence, for all $n \geq N$ and $t \geq 0$,

$$\big|F\big(t, u_n(t)\big) - F\big(t, u(t)\big)\big| < \epsilon. \tag{2.1.8}$$

Taking the supremum over all $t \geq 0$ shows that

$$\left\| \widetilde{F}(u_n) - \widetilde{F}(u) \right\|_\infty < \epsilon, \quad \forall n \geq N. \tag{2.1.9}$$

Thus $\widetilde{F}(u_n) \to \widetilde{F}(u)$ in $C_{\{0\}}$ as $n \to \infty$, which proves that $\widetilde{F}$ is continuous from $C^1_{\{0\}}$ to $C_{\{0\}}$.

It remains to prove that $\widetilde{F}$ is C^1, and to calculate its derivative. We first show differentiability directly, by verifying (2.1.4). Thus we have to show that for each $u \in C^1_{\{0\}}$,

$$\lim_{\|h\|_{1,\infty} \to 0} \frac{\widetilde{F}(u+h) - \widetilde{F}(u) - \widetilde{D_x F}(u)h}{\|h\|_{1,\infty}} = 0 \in C_{\{0\}}. \tag{2.1.10}$$

To begin, we must show that the above expression is well-defined, in the sense that the map $h \mapsto \widetilde{D_x F}(u)h$ is really a linear operator from $C^1_{\{0\}}$ into $C_{\{0\}}$. The linearity is obvious, as is the fact that $\widetilde{D_x F}(u)h$ is continuous on $[0, \infty)$. To show that in fact $\widetilde{D_x F}(u)h$ is in $C_{\{0\}}$, it suffices (since $h \in C_{\{0\}}$) to check that $\widetilde{D_x F}(u)$ is bounded. This boundedness follows from the compactness of $\overline{\operatorname{rge} u}$ via assumption (2.1.1b).

Now, to prove equation (2.1.10), we examine the $C_{\{0\}}$ norm (supremum norm) of the numerator of the difference quotient. For the moment, denote the numerator in (2.1.10) by $v \in C_{\{0\}}$. Unless $v = 0$ (in which case there is nothing to prove), we can find some $T > 0$ such that $|v(t)| < \|v\|_\infty / 2$ for $t > T$. Thus, $\|v\|_\infty$ is the same as the supremum of $|v(t)|$ on the compact set $[0, T]$. This shows that v attains its supremum as a maximum, and there is some least $t = t_h \geq 0$ such that

$$\left\| \widetilde{F}(u+h) - \widetilde{F}(u) - \widetilde{D_x F}(u)h \right\|_\infty = \left| \widetilde{F}(u+h)(t_h) - \widetilde{F}(u)(t_h) - \widetilde{D_x F}(u)(t_h)h(t_h) \right|. \tag{2.1.11}$$

By setting $\xi_h := u(t_h) \in \mathbb{R}^d$ and $\eta_h := h(t_h) \in \mathbb{R}^d$, we can then write (2.1.11) as

$$\left\| \widetilde{F}(u+h) - \widetilde{F}(u) - \widetilde{D_x F}(u)h \right\|_\infty = \left| F(t_h, \xi_h + \eta_h) - F(t_h, \xi_h) - D_x F(t_h, \xi_h)\eta_h \right|. \tag{2.1.12}$$

We write (2.1.12) in integral form:

$$
\begin{aligned}
\big|F(t_h,\, \xi_h+\eta_h) - F(t_h,\, \xi_h) - & D_xF(t_h,\, \xi_h)\eta_h\big| \\
&= \left|\int_0^1 \big[D_xF(t_h,\, \xi_h+s\eta_h) - D_xF(t_h,\, \xi_h)\big]\eta_h\,\mathrm{d}s\right| \qquad (2.1.13)\\
&\le \big|\eta_h\big| \int_0^1 \big|D_xF(t_h,\, \xi_h+s\eta_h) - D_xF(t_h,\, \xi_h)\big|\,\mathrm{d}s. \qquad (2.1.14)
\end{aligned}
$$

Because one has

$$
\frac{\big|\eta_h\big|}{\big\|h\big\|_{1,\,\infty}} \le \frac{\big|h(t_h)\big|}{\big\|h\big\|_{\infty}} \le 1, \tag{2.1.15}
$$

we have reduced the proof of (2.1.10) to the verification that

$$
\left(\int_0^1 \big|D_xF(t_h,\, \xi_h+s\eta_h) - D_xF(t_h,\, \xi_h)\big|\,\mathrm{d}s\right) \to 0 \quad \text{as} \quad h\to 0 \text{ in } C^1_{\{0\}}. \tag{2.1.16}
$$

Let K be a closed ball centered at the origin that contains the range of u. For sufficiently small $\big\|h\big\|_{1,\,\infty}$ (even sufficiently small $\big\|h\big\|_{\infty}$), the vector $\xi_h + s\eta_h = u(t_h) + sh(t_h)$ will lie in the compact subset $2K$ of $\mathbb{R}^d$, for all $0 \le s \le 1$. By assumption (2.1.1b), D_xF is uniformly continuous on $[0,\,\infty)\times 2K$. Noting once more that $\eta_h \le \big\|h\big\|_{1,\,\infty}$, the integrand of (2.1.16) can be made smaller than any preassigned $\epsilon > 0$, provided only that $\big\|h\big\|_{1,\,\infty}$ Is sufficiently small. This proves (2.1.10).

Finally, we are to show that $D\widetilde{F}(u)$ varies continuously with u. Fix $u \in C^1_{\{0\}}$ and $\epsilon > 0$. Let K be the closed ball of radius $\big\|u\big\|_{1,\,\infty}+1$ centered at the origin. By the assumed uniform continuity of D_xF on $[0,\,\infty)\times K$, there is a $\delta > 0$ such that $\big|D_xF(t, u(t)) - D_xF(t,\, x)\big| < \epsilon$

whenever $\left|u(t)-x\right|<\delta$. (As long as $\delta<1$, this condition automatically puts x in K.) Let $v\in C^1_{\{0\}}$ be such that $\left\|u-v\right\|_{1,\infty}<\delta$. Then, using the already proved (2.1.4), we have

$$\left\|D\widetilde{F}(u)-D\widetilde{F}(v)\right\|_{\mathrm{op}}=\sup\left\{\left\|\left[\widetilde{D_xF}(u)-\widetilde{D_xF}(v)\right]h\right\|_\infty : \left\|h\right\|_{1,\infty}\le 1\right\} \tag{2.1.17}$$

$$=\sup\left\{\left|\left[D_xF(t,\,u(t))-D_xF(t,\,v(t))\right]h(t)\right| : t\ge 0,\ \left\|h\right\|_{1,\infty}\le 1\right\} \tag{2.1.18}$$

$$\le\sup_{t\ge 0}\left|D_xF(t,\,u(t))-D_xF(t,\,v(t))\right| \tag{2.1.19}$$

$$<\epsilon, \tag{2.1.20}$$

since

$$\left|u(t)-v(t)\right|\le\left\|u-v\right\|_{1,\infty}<\delta. \tag{2.1.21}$$

This completes the proof. □

We now have enough information to know the following, as a corollary of Theorem 2.1.2.

Corollary 2.1.3. *Let P be a projection on $\mathbb{R}^d$. Assuming that $F:[0,\,\infty)\times\mathbb{R}^d\to\mathbb{R}^d$ satisfies conditions (2.1.1a), (2.1.1b), and (2.1.1c), the expression*

$$\Phi_{F,\,P}(u):=\big(\dot{u}+\widetilde{F}(u),\ Pu(0)\big),\quad \forall u\in C^1_{\{0\}} \tag{2.1.22}$$

defines a C^1 map of $C^1_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big)$ into $C_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big)\times\operatorname{rge}P$. The derivative of $\Phi_{F,\,P}$ at the point u is given by

$$D\Phi_{F,\,P}(u)h(t)=\Big(\dot{h}(t)+D_xF\big(t,\,u(t)\big)h(t),\ Ph(0)\Big)\quad\forall u,\,h\in C^1_{\{0\}},\quad \forall t\ge 0. \tag{2.1.23}$$

Proof. The map $u\mapsto\dot{u}$ is continuous and linear from $C^1_{\{0\}}$ into $C_{\{0\}}$. The map $u\mapsto Pu(0)$ is continuous and linear from $C^1_{\{0\}}$ into $\operatorname{rge}P$. Thus, this is a simple corollary of Theorem 2.1.2.

□

We formally state the definition that is justified by Corollary 2.1.3.

Definition 2.1.4. Let P be a projection on $\mathbb{R}^d$. Assume that $F : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies (2.1.1a) – (2.1.1c). Define $\Phi_{F,P} : C^1_{\{0\}}\big([0, \infty), \mathbb{R}^d\big) \to C_{\{0\}}\big([0, \infty), \mathbb{R}^d\big)$ by

$$\Phi_{F,P}(u) := \big(\dot{u} + \widetilde{F}(u), Pu(0)\big), \quad \forall u \in C^1_{\{0\}}. \tag{2.1.24}$$

The following remark is now clear.

Remark 2.1.5. Let $f \in C_{\{0\}}\big([0, \infty), \mathbb{R}^d\big)$. Let $F : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ satisfy (2.1.1a) – (2.1.1c). Then a continuously differentiable function $u : [0, \infty) \to \mathbb{R}^d$ satisfies

$$\begin{cases} \dot{u}(t) + F\big(t, u(t)\big) = f(t), & \forall t \geq 0, \\ Pu(0) = \xi, \\ \lim\limits_{t\to\infty} u(t) = 0 \end{cases} \tag{2.1.25}$$

if and only if $u \in C^1_{\{0\}}\big([0, \infty), \mathbb{R}^d\big)$ and $\Phi_{F,P}(u) = (f, \xi)$. $\Diamond$

2.1.2 Compact perturbations

Before finishing our study of the Nemytskii operator $\widetilde{F}$, we turn to a compactness issue that will arise later in Section 2.3, when we are looking at the Fredholm property. For this, we will use the following variant of Ascoli's Theorem, which can be found in the introductory remarks of the recent paper of Rabier [Rab04a].[2] The idea is to identify $C_{\{0\}}\big([0, \infty), \mathbb{R}^d\big)$ with the space of continuous functions on the interval $[0, 1]$ that vanish at 1. This is possible because the functions in $C_{\{0\}}\big([0, \infty), \mathbb{R}^d\big)$ all tend to zero at ∞. The result then follows by applying the classical Ascoli theorem, and then pulling back to $C_{\{0\}}\big([0, \infty), \mathbb{R}^d\big)$.

[2]We have specialized to the current situation ($E = \mathbb{R}^d$, $z = 0$, $\mathcal{H}$ is a sequence).

Lemma 2.1.6. *A sequence (f_n) drawn from $C_{\{0\}}\big([0, \infty), \mathbb{R}^d\big)$ has a convergent subsequence if and only if all three of the following conditions hold:*

1. *For all $t \geq 0$, the set $\{f_n(t)\colon n \in \mathbb{N}\}$ is bounded.*
2. *The sequence (f_n) is equicontinuous.*
3. *The convergence of $f_n(t)$ to zero as $t \to \infty$ occurs uniformly in n.*

Lemma 2.1.7. *Suppose that F satisfies (2.1.1a) – (2.1.1c). Then $D\widetilde{F}(u) - D\widetilde{F}(0)$ is a compact linear operator from $C^1_{\{0\}}\big([0, \infty), \mathbb{R}^d\big)$ into $C_{\{0\}}\big([0, \infty), \mathbb{R}^d\big)$, for each $u \in C^1_{\{0\}}$.*

Proof. Using (2.1.3) from Theorem 2.1.2 to express $D\widetilde{F}$, we have

$$\big(D\widetilde{F}(u) - D\widetilde{F}(0)\big)h = \big(D_xF(\cdot, u(\cdot))h(\cdot) - D_xF(\cdot, 0)h(\cdot), 0\big). \tag{2.1.26}$$

To show that this defines a compact linear operator on $C^1_{\{0\}}$, let (h_n) be a bounded sequence in $C^1_{\{0\}}$, and let

$$g_n(t) := \big(D_xF(t, u(t)) - D_xF(t, 0)\big)h_n(t), \quad \forall t \geq 0, \forall n \in \mathbb{N}. \tag{2.1.27}$$

We show that the sequence (g_n) admits a $C_{\{0\}}$-convergent subsequence. To do so, we use Lemma 2.1.6, above. For boundedness, note that the closure of the range of u is a compact subset K of $\mathbb{R}^d$ and that $0 \in K$. Hence, the assumed boundedness of D_xF on $[0, \infty) \times K$ (see (2.1.1b)) and the boundedness of the sequence (h_n) in $C^1_{\{0\}}$ imply the boundedness of the sequence (g_n) in $C^1_{\{0\}}$.

For equicontinuity, let $t_0 \geq 0$ be fixed. Let $M > 0$ be a bound for $\bigl|D_xF(t,\,x)\bigr|$ on $[0,\,\infty) \times K$. Let $N > 0$ be a bound for the sequence $\bigl(\|h_n\|_{1,\infty}\bigr)$. We use the triangle inequality to estimate that, for all $t \geq 0$,

$$
\begin{aligned}
&|g_n(t) - g_n(t_0)| \\
&\quad = \bigl|\bigl(D_xF(t,\,u(t)) - D_xF(t,\,0)\bigr)h_n(t) - \bigl(D_xF(t_0,\,u(t_0)) - D_xF(t_0,\,0)\bigr)h_n(t_0)\bigr| \qquad (2.1.28)\\
&\quad \leq \Bigl|\Bigl(D_xF\bigl(t,\,u(t)\bigr) - D_xF(t,\,0)\Bigr)\bigl((h_n(t) - h_n(t_0)\bigr)\Bigr| \\
&\qquad\quad + \Bigl|\Bigl(\bigl(D_xF(t_0,\,u(t_0)) - D_xF(t_0,\,0)\bigr) - \bigl(D_xF(t,\,u(t)) - D_xF(t,\,0)\bigr)\Bigr)h_n(t_0)\Bigr| \qquad (2.1.29)\\
&\quad \leq \Bigl|\Bigl(D_xF\bigl(t,\,u(t)\bigr) - D_xF(t,\,0)\Bigr)\bigl((h_n(t) - h_n(t_0)\bigr)\Bigr| \\
&\qquad\qquad + \Bigl|\Bigl(D_xF\bigl(t,\,u(t)\bigr) - D_xF\bigl(t_0,\,u(t_0)\bigr)\Bigr)h_n(t_0)\Bigr| \\
&\qquad\qquad\qquad + \Bigl|\Bigl(D_xF(t,\,0) - D_xF(t_0,\,0)\Bigr)h_n(t_0)\Bigr|. \qquad (2.1.30)
\end{aligned}
$$

Using the bounds M and N, this estimate implies that

$$
\begin{aligned}
&\bigl|g_n(s) - g_n(t_0)\bigr| \leq 2MN\,\bigl|t - t_0\bigr| \\
&\qquad + N\,\Bigl|\Bigl(D_xF\bigl(t,\,u(t)\bigr) - D_xF\bigl(t_0,\,u(t_0)\bigr)\Bigr)\Bigr| + N\,\Bigl|\Bigl(D_xF(t,\,0) - D_xF(t_0,\,0)\Bigr)\Bigr|, \qquad (2.1.31)
\end{aligned}
$$

where the uniform bound N on the sequence $\bigl(\|h_n\|_{1,\infty}\bigr)$ has been used in the first term as a Lipschitz constant. Now let $\epsilon > 0$. For control of the first term on the right side of (2.1.31), let $\delta_1 = \epsilon$. For the next term, we use the uniform continuity of D_xF on $[0,\,\infty) \times K$ to find $\delta_2 > 0$ such that $\bigl|D_x(t,\,\xi) - D_x(t_0,\eta)\bigr| < \epsilon$ whenever $\bigl|t - t_0\bigr| + \bigl|\xi - \eta\bigr| < \delta_2$ and $\xi,\,\eta \in K$. We then use the continuity of u at t_0 to find some $\delta_3 > 0$ such that $\bigl|u(t) - u(t_0)\bigr| < \delta_2/2$ whenever $\bigl|t - t_0\bigr| < \delta_3$. Put $\delta = \min(\delta_1,\,\delta_2/2,\,\delta_3)$. Then, as long as $\bigl|t - t_0\bigr| < \delta$,

$$
\bigl|g_n(t) - g_n(t_0)\bigr| < 2MN\epsilon + N\epsilon + N\epsilon, \quad \forall n \in \mathbb{N}, \qquad (2.1.32)
$$

which proves the equicontinuity of (g_n).

Finally, we must check that $g_n(t)$ tends n-uniformly to 0 as t tends to infinity. Given the boundedness of the sequence (h_n) in $C^1_{\{0\}}$, it suffices to show that

$$\lim_{t\to\infty}\big(D_xF(t,\,u(t)) - D_xF(t,\,0)\big) = 0 \text{ in } \mathbb{R}^{d\times d}. \tag{2.1.33}$$

For any $\epsilon > 0$, choose $\delta > 0$ so that if ξ is in the closure of the range of u and if $\big|\xi\big| < \delta_2$, then

$$\big|D_xF(t,\,\xi) - D_xF(t,\,0)\big| < \epsilon \tag{2.1.34}$$

for all $t \geq 0$. Once again, this is possible by (2.1.1b). Since $\big|u(t)\big| < \delta_2$ for sufficiently large t, this verifies (2.1.33). □

The following corollary of Lemma 2.1.7 will be of use in Section 2.3.1.

Corollary 2.1.8. *Assume that $F : [0,\,\infty)\times\mathbb{R}^d \to \mathbb{R}^d$ satisfies (2.1.1a) – (2.1.1c). Then the linear map $D\Phi(u) - D\Phi(0)$ is compact from $C^1_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big)$ into $C_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big)\times \operatorname{rge} P$, for each $u \in C^1_{\{0\}}$.*

Proof. According to Corollary 2.1.3, for all $h \in X$ and $t \geq 0$,

$$\big(D\Phi(u) - D\Phi(0)\big)h(t) = \Big(\Big[D_xF\big(t,\,u(t)\big) - D_xF\big(t,\,0\big)\Big]h(t),\,0\Big). \tag{2.1.35}$$

Hence, using equation (2.1.3) in Theorem 2.1.2 on page 17,

$$D\Phi(u) - D\Phi(0) = \big(D\widetilde{F}(u) - D\widetilde{F}(0),\,0\big), \tag{2.1.36}$$

the compactness of which follows from Lemma 2.1.7. □

2.2 PROPERNESS ON THE CLOSED BOUNDED SUBSETS

We seek conditions on F such that $\Phi_{F,P}$ will be proper[3] on the closed bounded subsets of $C^1_{\{0\}}\big([0,\infty),\mathbb{R}^d\big)$. In [RSb, RSa], Rabier and Stuart show that $\Phi_{F,P}\colon W^{1,p}\to L^p$ is proper on the closed bounded subsets of $W^{1,p}\big([0,\infty),\mathbb{R}^d\big)$ under the assumptions that (i) there is some $F^\infty\in C^1(\mathbb{R}^d,\mathbb{R}^d)$ such that $F(0)=0$ and $D_xF(t,x)\to DF^\infty(x)$ as $t\to\infty$, uniformly on bounded subsets of $\mathbb{R}^d$, and (ii) that $DF(0)$ has no imaginary eigenvalues and the equation $\dot{u}(t)+F^\infty\big(u(t)\big)=0$ has only the trivial solution in $W^{1,p}(\mathbb{R},\mathbb{R}^d)$. In this section, we prove a similar result in the continuous setting. We remove the first of the above two conditions, thus relaxing the assumptions about the asymptotic behavior of $F(t,x)$ as $t\to\infty$. We find that a condition similar to (ii) remains sufficient for properness, but with the function F^∞ replaced by a family of functions that captures enough of the asymptotic behavior of F.

2.2.1 Topological preliminaries

Definition 2.2.1. Let $F:[0,\infty)\times\mathbb{R}^d\to\mathbb{R}^d$ and $u:[0,\infty)\to\mathbb{R}^d$. For any $\sigma\in\mathbb{R}$, we define the σ-translates $\tau_\sigma F$ of F and $\tau_\sigma u$ of u by

$$\tau_\sigma F(t,x) := F(t+\sigma,x), \qquad \forall t\in[-\sigma,\infty),\ \forall x\in\mathbb{R}^d, \tag{2.2.1}$$

$$\tau_\sigma u(t) := u(t+\sigma), \qquad \forall t\in[-\sigma,\infty). \tag{2.2.2}$$

We need not limit the use of τ_σ to the above two examples. It will be understood that if J is any subinterval of $\mathbb{R}$, if S and T are sets, and if $f\colon J\times S\to T$ is a function, then $\tau_\sigma f\colon(J-\sigma)\times S\to T$ is defined by $\tau_\sigma f(t,s)=f(t+\sigma,s)$. ♦

[3] See Section 1.4.

We will have particular interest in the topological structure of the family of translates $\{\tau_\sigma F\}_{\sigma\geq 0}$.

Definition 2.2.2. Let a sequence $(\sigma_n)_{n\in\mathbb{N}}$ be given, such that $\lim_{n\to\infty}\sigma_n = +\infty$. We say that $G:\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^d$ is the CO-limit of the sequence $\left(\tau_{\sigma_n}F\right)$ of translates of F if if $\left(\tau_{\sigma_n}F\right)$ converges uniformly to G on each compact subset of $\mathbb{R}\times\mathbb{R}^d$ as $n\to\infty$ (see Remark 2.2.3, below). In this case, we write

$$G = \underset{n\to\infty}{\text{CO-lim}}\, G_n. \tag{2.2.3}$$

We denote the collection of all such G by $\omega(F)$, which we call the "omega-limit set of F".

Remark 2.2.3. In the above definition, note that $F(t+\sigma_n,\, x)$ is defined only when $t\geq -\sigma_n$. Hence, by "convergence of $\left(\tau_{\sigma_n}F\right)$ to G" on a given compact set, we mean convergence of any tail of the sequence $\left(\tau_{\sigma_n}F\right)$which is defined on that compact set. $\Diamond$

Definition 2.2.4. We denote by $\omega_0(F)\subset\mathbb{R}^d$ the following set:

$$\omega_0(F) := \left\{x\in\mathbb{R}^d\colon \exists G\in\omega(F) \text{ such that } \forall t\in\mathbb{R},\, G(t,\, x)=0\right\}. \tag{2.2.4}$$

It is not obvious that the set $\omega(F)$ is nonempty. This situation is clarified by the following lemma, which states that the set of all positive translates of F is sequentially relatively compact in the topology of convergence on compact sets.[4]

Lemma 2.2.5. *Suppose that for each compact subset K of $\mathbb{R}^d$, the function $F:[0,\,\infty)\times\mathbb{R}^d\to\mathbb{R}^d$ is bounded and uniformly continuous on $[0,\,\infty)\times K$. Then for each sequence $(\sigma_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\sigma_n = +\infty$, there is a subsequence (σ_{n_k}) and a function $G:\mathbb{R}\times\mathbb{R}^d$ such that*

$$G = \underset{k\to\infty}{\text{CO-lim}}\, \tau_{\sigma_{n_k}}F. \tag{2.2.5}$$

[4]In fact, this is equivalent to relative compactness, because the underlying topology is a metric topology.

Proof. Let $J_1 = [-1,\, 1] \subset \mathbb{R}$ and let $K_1 = \{\xi \in \mathbb{R}^d \mid \left|\xi\right| \leq 1\}$. Since we are seeking a subsequence, we may begin by supposing that $\sigma_n \geq 1$ for all $n \in \mathbb{N}$, so that $\tau_{\sigma_n} F$ is defined on $J_1 \times K_1$. We will apply the classical Ascoli theorem to the sequence of restrictions of $\tau_{\sigma_n}(F)$ to the compact set $J_1 \times K_1$. Equicontinuity follows from the assumed uniform continuity of F on $[0,\, \infty) \times K_1$, since

$$\left|\tau_{\sigma_n} F(t,\, x) - \tau_{\sigma_n} F(s,\, y)\right| = \left|F(t + \sigma_n,\, x) - F(s + \sigma_n,\, y)\right|, \tag{2.2.6}$$

and

$$\left|(t + \sigma_n) - (s + \sigma_n)\right| = \left|t - s\right|. \tag{2.2.7}$$

Boundedness follows from the assumed boundedness of F on $[0,\, \infty) \times\ K_1$. So, by the Ascoli Theorem, we can extract a subsequence $\tau_{\sigma_{(1,\, n)}}(F)$ whose restrictions to $J_1 \times K_1$ converge uniformly to a continuous function G_1 on $J_1 \times K_1$. By starting this subsequence at a higher rank, if necessary, we ensure that $\sigma_{1,\, n} \geq 2$ for all $n \in \mathbb{N}$. This completes the base step of an inductive construction. To avoid a notational difficulty, let $\left(\sigma_{(0,\, n)}\right)$ denote the original sequence (σ_n).

Suppose now that for some $k \in \mathbb{N}$ we have extracted a subsequence $\left(\sigma_{(k,\, n)}\right)_{n \in \mathbb{N}}$ of $\left(\sigma_{(k-1,\, n)}\right)_{n \in \mathbb{N}}$ such that the restrictions of $\tau_{\sigma_{(k,\, n)}}(F)$ to the compact set $J_k \times K_k$ converge uniformly to a continuous function G_k on $J_k \times K_k$. Here, $J_k = [-k,\, k]$ and K_k is the closed ball of radius k in $\mathbb{R}^d$. Suppose also that $\sigma_{(k,\, n)} \geq k + 1$ for all $n \in \mathbb{N}$. This is exactly what we had already done for $k = 1$.

We apply the Ascoli theorem to the restrictions of the sequence $\tau_{\sigma_{(k,\, n)}}(F)$ to the compact set $J_{k+1} \times K_{k+1}$ to obtain a further subsequence $\tau_{\sigma_{(k+1,\, n)}}(F)$ whose restrictions to the set $J_{k+1} \times K_{k+1}$ converge uniformly to a continuous function G_{k+1} on $J_{k+1} \times K_{k+1}$. By starting this subsequence at a higher rank, if necessary, we ensure that $\sigma_{(k+1,\,n)} \geq k+2$ for all $n \in \mathbb{N}$. We continue this procedure inductively for all $k \in \mathbb{N}$.

Notice that G_{k+1} extends G_k, since the passage to a subsequence did not change any of the already determined limiting values in $J_k \times K_k$. We can therefore define a continuous function G by

$$G(t,\, x) := G_k(t,\, x), \quad \text{where } k \geq \left|t\right|. \tag{2.2.8}$$

We claim that the diagonal sequence $(\sigma_{(n,\,n)})$ is the sought subsequence, and moreover that $(\tau_{\sigma_{(n,\,n)}} F)$ has CO-limit G. Fix a compact subset S of $\mathbb{R} \times \mathbb{R}^d$. For large enough k, S is contained in a set $J_k \times K_k$ encountered during the above construction. Hence the k^{th} defined subsequence $\big(\tau_{\sigma_{(k,\,n)}}(F)\big)$ converges uniformly to G_k on S. But the k-tail of the diagonal sequence $\big(\tau_{\sigma_{(n,\,n)}} F\big)$ is a subsequence of $\big(\tau_{\sigma_{(k,\,n)}}(F)\big)$, and hence also converges uniformly to G_k on S. Since G_k agrees with G on S, this completes the proof. □

Corollary 2.2.6. *For each compact subset K of $\mathbb{R}^d$, suppose that $F : [0,\, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is bounded and uniformly continuous on $[0,\, \infty) \times K$. Then the set $\omega(F)$ is nonempty.*

Proof. For any sequence tending to infinity, say $\sigma_n = n$, the function G whose existence is asserted by Lemma 2.2.5 is an element of $\omega(F)$. □

Next we show that the limiting process preserves the relevant boundedness and uniform continuity properties.

Lemma 2.2.7. *Suppose once again that $F : [0,\, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is both bounded and uniformly continuous on $\mathbb{R} \times K$, for each compact subset K of $\mathbb{R}^d$. Then, for each $G \in \omega(F)$ and each compact $K \subset \mathbb{R}^d$, G is bounded and uniformly continuous on $\mathbb{R} \times K$.*

Proof. Let $G \in \omega(F)$ and compact $K \subset \mathbb{R}^d$ be given. Since $G \in \omega(F)$, there is a sequence $\sigma_n \to \infty$ such that

$$G = \operatorname*{CO-lim}_{n\to\infty} \tau_{\sigma_n} F. \tag{2.2.9}$$

The boundedness of G on $\mathbb{R} \times K$ follows directly from the boundedness of F on $[0,\, \infty) \times K$. For the uniform continuity, let $\epsilon > 0$ and let $\delta > 0$ be such that whenever $s, t \in [0,\, \infty)$ and $x, y \in K$ and $\big|s - t\big| + \big|x - y\big| < \delta$ we have $\big|F(s,\, x) - F(t,\, y)\big| < \epsilon$. This choice of δ is possible because of the assumed uniform continuity of F on $[0,\, \infty) \times K$. Now take $s, t \in \mathbb{R}$ and $x, y \in K$ such that $\big|s - t\big| + \big|x - y\big| < \delta$. As soon as we show that $\big|G(s,\, x) - G(t,\, y)\big| < 3\epsilon$, we are done.

There is $n = n(s, t) \in \mathbb{N}$ such that

$$\big|G(s,\, x) - F(s + \sigma_n,\, x)\big| + \big|G(t,\, y) - F(t + \sigma_n,\, y)\big| < 2\epsilon. \tag{2.2.10}$$

This is just the convergence of $\tau_{\sigma_n}(F)$ to G at the points $(s,\, x)$ and $(t,\, y)$. Hence, the triangle inequality gives

$$\begin{aligned} \big|G(s,\, x) - G(t,\, y)\big| \le \big|G(s,\, x) - F(s + \sigma_n,\, x)\big| + \big|G(t,\, y) - F(t + \sigma_n,\, y)\big| \\ + \big|F(s + \sigma_n,\, x) - F(t + \sigma_n,\, y)\big| < 3\epsilon, \end{aligned} \tag{2.2.11}$$

as long as $s, t \in \mathbb{R}$ and $x, y \in K$ are such that

$$\big|(s + t_n) - (t + t_n)\big| + \big|x - y\big| = \big|s - t\big| + \big|x - y\big| < \delta. \tag{2.2.12}$$

This proves the uniform continuity of G on $\mathbb{R} \times K$. $\square$

Lemma 2.2.8. *Suppose that $F : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is both bounded and uniformly continuous on $[0, \infty) \times K$, for each compact subset K of $\mathbb{R}^d$. Then the Nemytskii operator $\widetilde{F}$ maps $C_b\big([0, \infty), \mathbb{R}^d\big)$ continuously into itself. If also $\lim_{t\to\infty} F(t, 0) = 0$, then $\widetilde{F}$ maps $C_{\{0\}}\big([0, \infty), \mathbb{R}^d\big)$ (continuously) into itself.*

Proof. Let $u \in C_b$. Since u is continuous, the map $t \mapsto (t, u(t)) \in [0, \infty) \times \mathbb{R}^d$ is also continuous. The composition $\widetilde{F}(u)$ is therefore also continuous. To see that $\widetilde{F}(u)$ is bounded, let $K = \overline{\operatorname{rge} u} \subset \mathbb{R}^d$. Since u is bounded, K is compact. The boundedness of F on $[0, \infty) \times K$ ensures that $\widetilde{F}(u)$ is bounded, since $\widetilde{F}(u) = F(t, u(t))$ and $(t, u(t)) \subset [0, \infty) \times K$. Thus, $\widetilde{F}(u) \in C_b$.

Next, to see that $\widetilde{F}$ is continuous on C_b, let (u_n) be a sequence that converges in C_b to a function u. Hence, $\{u_n\colon n \in \mathbb{N}\} \cup \{u\}$ is a norm-bounded subset of C_b. We can thus find a compact subset K of $\mathbb{R}^d$ that contains the ranges of all of the functions u_n and u. Now we use the assumed uniform continuity of F on $[0, \infty) \times K$. Let $\epsilon > 0$; there exists $\delta > 0$ such that $\big|F(s, x) - F(t, y)\big| < \epsilon$ for all $s, t \in [0, \infty)$ and $x, y \in K$ such that $\big|s - t\big| + \big|x - y\big| < \delta$. By the uniform convergence of (u_n), for all sufficiently large n, we have that $\big|u_n(t) - u(t)\big| < \delta$. Therefore, for such large values for n,

$$\Big|\widetilde{F}(u_n)(t) - \widetilde{F}(u)(t)\Big| = \big|F(t, u_n(t)) - F(t, u(t))\big| < \epsilon. \tag{2.2.13}$$

Since this estimate is independent of $t \geq 0$, this shows that the sequence $\big(\widetilde{F}(u_n)\big)$ converges to $\widetilde{F}(u)$ in C_b. This proves the first assertion.

To prove the second assertion, we need only show that

$$\lim_{t\to\infty} F\big(t, u(t)\big) = 0 \tag{2.2.14}$$

whenever $u \in C_{\{0\}}$ and $F(t,\,0) \to 0$ as $t \to \infty$. Let $\epsilon > 0$. Since u is bounded, there is a compact set K which contains the range of u in $\mathbb{R}^d$. By the uniform continuity of F on $[0,\,\infty) \times K$, along with the convergence of $u(t)$ to $0 \in K$ as $t \to \infty$, there is some $T > 0$ such that

$$\big|F\big(t,\,u(t)\big) - F(t,\,0)\big| < \epsilon/2, \quad t \geq T. \tag{2.2.15}$$

Also, by increasing T if necessary, the assumption that $\lim_{t\to\infty} F(t,\,0) = 0$ ensures that

$$\big|F(t,\,0)\big| < \epsilon/2, \quad t \geq T. \tag{2.2.16}$$

Together, we have that $\big|F\big(t,\,u(t)\big)\big| < \epsilon$ if $t > T$, which proves (2.2.14). $\square$

Lemma 2.2.9. *Suppose that $F : [0,\,\infty)\times\mathbb{R}^d \to \mathbb{R}^d$ is both bounded and uniformly continuous on $[0,\,\infty) \times K$, for each compact subset K of $\mathbb{R}^d$. Then the set $\omega_0(F)$ defined in (2.2.4) is closed.*

Proof. Let (x_n) be a sequence drawn from $\omega_0(F)$ that converges to some $x_0 \in \mathbb{R}^d$. By definition of $\omega_0(F)$, there are functions $G^{(n)} \in \omega(F)$ such that

$$G^{(n)}(t,\,x_n) = 0, \quad \forall t \in \mathbb{R},\, \forall n \in \mathbb{N}. \tag{2.2.17}$$

Correspondingly, to each $n \in \mathbb{N}$ there corresponds a sequence $(\sigma_m^{(n)})$ in $[0,\,\infty)$ such that

$$G^{(n)} = \underset{m\to\infty}{\text{co-lim}}\, \tau_{\sigma_m^{(n)}} F. \tag{2.2.18}$$

We now construct a sequence (s_N) in $[0,\,\infty)$ such that both

$$\left.\begin{aligned} \lim_{N\to\infty} F(t + s_N,\,x_0) = 0, \qquad & \forall t \in \mathbb{R}, \\ \lim_{N\to\infty} s_N = \infty. \qquad & \end{aligned}\right\} \tag{2.2.19}$$

Take any $N \in \mathbb{N}$. Let $K := \{x_n \colon n \in \mathbb{N}\} \cup \{x_0\}$. By the assumed uniform continuity of F on $[0, \infty) \times K$ and the convergence of x_n to x_0, there is $n = n(N) \in \mathbb{N}$ such that

$$\big|F\big(t,\, x_0\big) - F\big(t,\, x_n\big)\big| < 1/N, \quad \forall t \geq 0. \tag{2.2.20}$$

For this choice of n, we use (2.2.18) with respect to the compact set $[N,\, N] \times K$, together with (2.2.17), to find $m = m(n, N) = m(N) \in \mathbb{N}$ such that

$$\big|F\big(t + \sigma_m^{(n)},\, x_n\big)\big| < 1/N, \quad \forall t \in [-N,\, N]. \tag{2.2.21}$$

It is implicit that m is large enough that $\sigma_m^{(n)} > N$. We now have

$$\begin{aligned}\big|F\big(t + \sigma_m^{(n)},\, x_0\big)\big| &\leq \big|F\big(t + \sigma_m^{(n)},\, x_0\big) - F\big(t + \sigma_m^{(n)},\, x_n\big)\big| + \big|F\big(t + \sigma_m^{(n)},\, x_n\big)\big| &(2.2.22)\\ &< 2/N, \quad \forall t \in [-N,\, N]. &(2.2.23)\end{aligned}$$

Take $s_N := \sigma_m^{(n)}$. (Recall that n and m were chosen in a way depending on N.) Hence

$$\left.\begin{aligned}&\big|F(t + s_N,\, x_0)\big| < 2/N, \qquad \forall t \in [-N,\, N],\\ &\lim_{N\to\infty} s_N = \infty.\end{aligned}\right\} \tag{2.2.24}$$

From this it is clear that the desired (2.2.19) follows. Finally, we use Lemma 2.2.5 to pass to a subsequence, again denoted by (s_N), such that

$$\operatorname*{CO-lim}_{N\to\infty} \tau_{s_N} F = G \tag{2.2.25}$$

for some $G \in \omega(F)$. It is immediate from (2.2.24) that $G(t,\, x_0) = 0$ for all $t \in \mathbb{R}$. Hence, $x_0 \in \omega_0(F)$, and we have proved that $\omega_0(F)$ is closed in $\mathbb{R}^d$, as desired. $\square$

Lemma 2.2.10. *Let J be any subinterval of $\mathbb{R}$. Suppose that (v_n) is a sequence in $C_b^1(J, \mathbb{R}^d)$ that converges uniformly on compact sets to a function $v \in C_b(J, \mathbb{R}^d)$. Suppose also that the sequence $(\dot{v}_n)$ converges uniformly on compact sets to $w \in C_b$. Then v is differentiable and $\dot{v} = w$.*

Proof. Fix a compact interval $K \subset J$, and choose $a \in K$. For any $t \in K$, and all $n \in \mathbb{N}$, we have the following instance of the Fundamental Theorem of Calculus:

$$v_n(t) = v(a) + \int_a^t \dot{v}_n(s)\,\mathrm{d}s. \tag{2.2.26}$$

Using the uniform convergence of $\dot{v}_n$ on C, we find that

$$\begin{aligned} v(t) = \lim_{n\to\infty} v_n(t) &= \lim_{n\to\infty} \left(v(a) + \int_a^t \dot{v}_n(s)\,\mathrm{d}s \right) && (2.2.27)\\ &= v(a) + \int_a^t \lim_{n\to\infty} \dot{v}_n(s)\,\mathrm{d}s && (2.2.28)\\ &= v(a) + \int_a^t w(s)\,\mathrm{d}s. && (2.2.29) \end{aligned}$$

Since w is continuous, both sides are differentiable and so

$$\dot{v}(t) = w(t) \tag{2.2.30}$$

follows from the Fundamental Theorem of Calculus. □

2.2.2 A characterization of relative compactness

The following characterization of compactness in $C_{\{0\}}(\mathbb{R}, \mathbb{R}^d)$ is found as Corollary 7 in Rabier's paper on Ascoli's Theorem [Rab04a]. In this statement, we have specialized the result to the current situation ($z = 0$, $E = \mathbb{R}^d$, δ-net $= \mathbb{R}$). Recall that a subset Z of $\mathbb{R}^d$ is *totally disconnected* if for each pair of distinct points $a \neq b$ in Z, there are open neighborhoods U_a and U_b of a and b (respectively) such that $Z \subset U_a \cup U_b$ and $Z \cap U_a \cap U_b = \emptyset$. Examples include finite sets, convergent sequences and their limits, Cantor sets, etc.

Lemma 2.2.11. *For a subset $\mathcal{H} \subset C_{\{0\}}(\mathbb{R}, \mathbb{R}^d)$, the following two statements are equivalent:*

1. *$\mathcal{H}$ is relatively compact in $C_{\{0\}}(\mathbb{R}, \mathbb{R}^d)$.*
2. *The following three statements hold:*
   a. *$\mathcal{H}(\mathbb{R})$ is bounded in $\mathbb{R}^d$.*
   b. *$\mathcal{H}$ is uniformly equicontinuous.*
   c. *There is a closed[5] and totally disconnected subset Z of $\mathbb{R}^d$ with the following property: If $u \in C_b(\mathbb{R}, \mathbb{R}^d)$ and there are sequences $(u_n) \subset \mathcal{H}$ and $(\sigma_n) \subset \mathbb{R}$ such that $\lim_{n\to\infty} |\sigma_n| = \infty$ and $\tau_{\sigma_n} u_n \to u$ pointwise on $\mathbb{R}$, then $u(\mathbb{R}) \subset Z$.*

We now verify that for the half-line setting, a similar result holds as a corollary:

Corollary 2.2.12. *For a subset $\mathcal{H} \subset C_{\{0\}}([0, \infty), \mathbb{R}^d)$, the following two statements are equivalent:*

1. *$\mathcal{H}$ is relatively compact in $C_{\{0\}}([0, \infty), \mathbb{R}^d)$.*
2. *The following three statements hold:*

[5] Since $\mathbb{R}^d$ is locally compact, we have taken advantage of the footnote to Corollary 6 in [Rab04a] to weaken the compactness to closedness.

a. $\mathcal{H}([0, \infty))$ *is bounded in* $\mathbb{R}^d$.

b. $\mathcal{H}$ *is uniformly equicontinuous.*

c. There is a closed and totally disconnected subset Z *of* $\mathbb{R}^d$ *with the following property: If* $u \in C_b(\mathbb{R}, \mathbb{R}^d)$ *and there are sequences* $(u_n) \subset \mathcal{H}$ *and* $(\sigma_n) \subset [0, \infty)$ *with* $\lim_{n\to\infty} \sigma_n = \infty$ *and such that* $\tau_{\sigma_n} u_n \to u$ *pointwise on* $\mathbb{R}$, *then* $u(\mathbb{R}) \subset Z$.

Proof. We define an extension operator $E : C_{\{0\}}\big([0, \infty)\big) \to C_{\{0\}}(\mathbb{R})$ as follows:

$$Eu(t) := \begin{cases} u(t) & \text{if } t \geq 0, \\ (1+t)u(t) & \text{if } -1 \leq t \leq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.31}$$

It is clear that E is a linear isometry of $C_{\{0\}}\big([0, \infty)\big)$ onto a closed subspace of $C_{\{0\}}(\mathbb{R})$. Hence a subset $\mathcal{H}$ of $C_{\{0\}}\big([0, \infty)\big)$ is relatively compact in $C_{\{0\}}\big([0, \infty)\big)$ if and only if its image $E\mathcal{H}$ is relatively compact in $C_{\{0\}}(\mathbb{R})$. That is to say, item 1 of Lemma 2.2.11 holds of $E\mathcal{H}$ if and only if item 1 of the present corollary holds of $\mathcal{H}$.

Hence, it suffices to show that item 2 of the present corollary holds for $\mathcal{H}$ if and only if the corresponding item 2 of Lemma 2.2.11 holds of $E\mathcal{H}$. This equivalence is clear for items 2 a and 2 b. It is also clear that if item 2 c in Lemma 2.2.11 holds for $E\mathcal{H}$, then the corresponding item 2 c of the present corollary must hold for $\mathcal{H}$, and with the same set Z. Indeed, this involves a restriction of our attention to sequences (σ_n) that tend to $+\infty$.

For the remaining implication, we suppose that item 2 c of the present corollary holds (for $\mathcal{H}$) and some Z. Note that by consideration of a constant sequence $u_n = u_0$, it must be that $0 \in Z$. Since every member of $E\mathcal{H}$ vanishes on $(-\infty, -1]$, the desired property holds easily for any sequence (σ_n) that has a subsequence tending to $-\infty$, for then the only possible pointwise limit of $\tau_{\sigma_n} E u_n$ is $0 \in Z$. For any other sequence (σ_n), one has $\sigma_n \to \infty$. Let $t \in \mathbb{R}$. For n sufficiently large that $\sigma_n > -t$, we have $E u_n(t + \sigma_n) = u_n(t + \sigma_n)$. Hence, the pointwise convergence of $\tau_{\sigma_n} E u_n$ to u implies the convergence of $\tau_{\sigma_n} u_n$ to u. Thus, $u(\mathbb{R}) \subset Z$ by assumption. This shows that item 2 c holds for $E\mathcal{H}$, and the proof is complete. $\square$

2.2.3 A sufficient condition for properness

To help streamline the statements of the following and later results, we introduce the following terminology.

Definition 2.2.13. As usual, suppose that for all compact subsets K of $\mathbb{R}^d$, the function $F : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is bounded and uniformly continuous on $[0, \infty) \times K$. We say that F has an *admissible* omega-set $\omega(F)$ if the following two conditions are satisfied:

1. The set $\omega_0(F) \subset \mathbb{R}^d$, as defined in (2.2.4), is totally disconnected.
2. If $G \in \omega(F)$ and u is a bounded C^1 solution to $\dot{u} + \widetilde{G}(u) = 0$ on all of $\mathbb{R}$, then u is constant. ♦

Theorem 2.2.14. *Once again, suppose that for all compact subsets K of $\mathbb{R}^d$, the function $F : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is bounded and uniformly continuous on $[0, \infty) \times K$. Let P be any projection on $\mathbb{R}^d$. If F has an admissible omega-set $\omega(F)$, it follows that the operator $\Phi_{F,P} : C^1_{\{0\}}\big([0, \infty), \mathbb{R}^d\big) \to C_{\{0\}}\big([0, \infty), \mathbb{R}^d\big) \times \operatorname{rge} P$ defined in Definition 2.1.4 is proper on the closed bounded subsets of $C^1_{\{0\}}\big([0, \infty), \mathbb{R}^d\big)$.*

Proof. Let $(u_n) \subset C^1_{\{0\}}\big([0, \infty)\big)$ be a bounded sequence. Suppose that $\big(\Phi_{F,P}(u_n)\big)$ is convergent. For the desired properness property, it suffices to show that (u_n) has a norm convergent subsequence in $C^1_{\{0\}}$. Hence, the only relevant information[6] in the supposed convergence of $(\Phi_{F,\,P}(u_n))$ is that $f_n := \dot{u}_n + \widetilde{F}(u_n)$ converges to some $f \in C_{\{0\}}$ as $n \to \infty$. We proceed by first establishing the relative compactness of $\mathcal{H} := \{u_n \mid n \in \mathbb{N}\}$ in $C_{\{0\}}$. We will then show that the resulting subsequence of (u_n) that converges in $C_{\{0\}}$ must also converge in $C^1_{\{0\}}$.

To prove that $\mathcal{H}$ is relatively compact in $C_{\{0\}}$, we will use Corollary 2.2.12. Notice first that $\mathcal{H}([0,\,\infty))$ is bounded. This is true because the sequence (u_n) is bounded in $C^1_{\{0\}}$. Next, for the uniform equicontinuity of $\mathcal{H}$, let M be a uniform bound for the $C^1_{\{0\}}$ norms of the functions (u_n). Then, for all $s,\,t \geq 0$ and all $n \in \mathbb{N}$,

$$\left|u_n(s) - u_n(t)\right| \leq M\left|s - t\right|, \tag{2.2.32}$$

which proves the uniform equicontinuity of $\mathcal{H}$.

To complete the application of Corollary 2.2.12, we set $Z = \omega_0(F)$. Note that this set is totally disconnected by assumption, and is closed because of Lemma 2.2.9. We choose a sequence from $\mathcal{H}$. It suffices to suppose that we have selected a subsequence of (u_n), which we again denote by (u_n). Let (σ_n) be a sequence in $[0,\,\infty)$ such that $\sigma_n \to \infty$. Supposing that $v_n := \tau_{\sigma_n}(u_n)$ converges pointwise on $\mathbb{R}$ to some $v \in C_b(\mathbb{R})$, we must show that $v(\mathbb{R}) \subset \omega_0(F)$.

To do this, we will show that v must be constant and that this constant must be in $\omega_0(F)$. Put $G_n := \tau_{\sigma_n}(F)$. We calculate that

$$\begin{aligned}
\dot{v}_n(t) + \widetilde{G_n}(v_n)(t) &= \tau_{\sigma_n}(\dot{u}_n)(t) + \big(\tau_{\sigma_n}(F)\big)(t,\,\tau_{\sigma_n}(u_n)(t)) && (2.2.33)\\
&= \dot{u}_n(t+\sigma_n) + F(t+\sigma_n,\,u_n(t+\sigma_n)) && (2.2.34)\\
&= f_n(t+\sigma_n) && (2.2.35)\\
&= \tau_{\sigma_n}(f_n)(t). && (2.2.36)
\end{aligned}$$

[6]The bounded sequence $(Pu_n(0))$ has a convergent subsequence without regard to the convergence of $((\Phi_{F,\,P}(u_n)))$.

so that

$$\dot{v}_n = -\widetilde{G_n}(v_n) + \tau_{\sigma_n}(f_n). \tag{2.2.37}$$

Now the boundedness of (v_n) in $C^1_{\{0\}}([-\sigma_n, \infty))$ implies that it possesses a subsequence that converges uniformly on compact sets to v. Indeed, the sequence is uniformly bounded, and equicontinuity follows as in (2.2.32), above. Hence, we can apply Ascoli's Theorem via a diagonal argument as in the proof of Lemma 2.2.5. This extracts the desired subsequence, again denoted (v_n).

Likewise, according to Lemma 2.2.5, there is a further subsequence of $G_n = \tau_{\sigma_n}(F)$ (again denoted by G_n) and some $G \in \omega(F)$ such that G_n converges to G uniformly on compact subsets of $\mathbb{R} \times \mathbb{R}^d$. We claim that the sequence $\big(\widetilde{G}_n(v_n)\big)$ therefore converges uniformly to $\widetilde{G}(v)$ on each compact subset of $\mathbb{R}$. Indeed, let J be any compact subset of $\mathbb{R}$, and let K be a compact subset of $\mathbb{R}^d$ that contains the ranges of the functions v_n (and v). Observe the following instance of the triangle inequality:

$$\left|\widetilde{G}_n(v_n)(t) - \widetilde{G}(v)(t)\right| \leq \left|G_n\big(t,\, v_n(t)\big) - G\big(t,\, v_n(t)\big)\right| + \left|G\big(t,\, v_n(t)\big) - G\big(t,\, v(t)\big)\right|. \tag{2.2.38}$$

The first term on the right side of (2.2.38) converges to zero uniformly on J because G_n converges to G uniformly on $J \times K$. The second term converges to zero uniformly on J because v_n converges uniformly to v on J, and G is uniformly continuous on $J \times K$. (See Lemma 2.2.7)

This explains half of what happens to (2.2.37) as $n \to \infty$. As for the other term on the right side of (2.2.37), note that

$$\left|\tau_{\sigma_n}(f_n)(t)\right| = \left|f_n(t+\sigma_n)\right| \leq \left|f_n(t+\sigma_n) - f(t+\sigma_n)\right| + \left|f(t+\sigma_n)\right|, \tag{2.2.39}$$

which shows that the sequence $\big(\tau_{\sigma_n}(f_n)\big)$ is convergent to zero on compact sets. Altogether, letting $n \to \infty$ in (2.2.37), we find that the derivatives $\dot{v}_n$ converge, uniformly on compact intervals, to $-\widetilde{G}(v)$. According to Lemma 2.2.10, this implies that v is differentiable and that

$$\dot{v} + \widetilde{G}(v) = 0. \tag{2.2.40}$$

Since $v \in C_b(\mathbb{R}, \mathbb{R}^d)$, it readily follows from Lemmas 2.2.7 and 2.2.8 that $\widetilde{G}(v) \in C_b$, and hence that $v \in C_b^1$. Hence, v is a constant c, by our main assumption. As a result, for all $t \in \mathbb{R}$, one has $G(t, c) = G(t, v(t)) = \dot{v}(t) = 0$. Thus, $v(\mathbb{R}) \in \omega_0(F)$ by definition. This completes the application of Corollary 2.2.12, from which it follows that the sequence (u_n) is relatively compact in $C_{\{0\}}$. Hence there exists a uniformly convergent subsequence, again denoted by (u_n). Call the limit u.

We need convergence in $C^1_{\{0\}}$. By Lemma 2.2.8, we have $\widetilde{F}(u_n) \to \widetilde{F}(u)$ in $C_{\{0\}}$. Recall that (f_n) is assumed to converge to f uniformly on $[0, \infty)$. Hence, $\dot{u}_n = \widetilde{F}(u_n) + f_n$ converges to $\widetilde{F}(u) + f$ uniformly on $[0, \infty)$. Hence, by Lemma 2.2.10, we have then that $\dot{u} = \widetilde{F}(u) + f$ and $\dot{u}_n \to \dot{u}$ in $C_{\{0\}}$. Thus $u_n \to u$ in $C^1_{\{0\}}$. □

2.2.4 Calculation of omega-limit sets

Here we explain how to calculate $\omega(F)$ in some cases.

Example 2.2.15. Suppose that for all compact subsets K of $\mathbb{R}^d$, the function $F : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is bounded and uniformly continuous on $[0, \infty) \times K$. The following two conditions are equivalent:

1. The following pointwise limit exists:

$$\lim_{t\to\infty} F(t,\, x) =: F^{\infty}(x), \quad \forall x \in \mathbb{R}^d. \tag{2.2.41}$$

2. The set $\omega(F)$ is the singleton $\{G\}$, where $G(t,\, x) = F^{\infty}(x)$ is autonomous.

Justification: According to Corollary 2.2.6, the set $\omega(F)$ is never empty. Hence, it suffices for the forward implication to show that $\omega(F) \subset \{G\}$. Let $H \in \omega(F)$, and choose $x \in \mathbb{R}^d$. Also choose $t \in \mathbb{R}$. Since $H \in \omega(F)$, there is a sequence (σ_n) such that $\sigma_n \to \infty$ and

$$\lim_{n\to\infty} F(t+\sigma_n,\, x) = H(t,\, x). \tag{2.2.42}$$

But (2.2.41) implies that

$$\lim_{n\to\infty} F(t+\sigma_n,\, x) = F^{\infty}(x). \tag{2.2.43}$$

Hence, $H(t,\, x) = F^{\infty}(x)$, as desired.

For the reverse implication, we suppose for contradiction that there is a point $x \in \mathbb{R}^d$ such that $F(t,\, x)$ does not converge to $F^{\infty}(x)$ as $t \to \infty$. In this case, there is a sequence $t_n \to \infty$ and $\epsilon_0 > 0$ such that

$$\left|F(t_n,\, x) - F^{\infty}(x)\right| > \epsilon_0 \tag{2.2.44}$$

for all $n \in \mathbb{N}$. But, according to Lemma 2.2.5, we can pass to a subsequence (t_{n_k}) such that

$$\operatorname*{CO-lim}_{k\to\infty} \tau_{t_{n_k}} F = H, \tag{2.2.45}$$

for some $H \in \omega(F)$. By assumption on $\omega(F)$, $H = G$. Specializing to the point $(0,\, x)$, this implies that

$$\lim_{k\to\infty} F(t_{n_k},\, x) = G(t,\, x) = F^{\infty}(x), \tag{2.2.46}$$

in contradiction with(2.2.44) ⧫

A special case of the above situation is that $F(t,\, x) = F(x)$ is itself autonomous.

Example 2.2.16. If $F(t, x) = F(x)$ is independent of t and is continuous, then $\omega(F)$ consists only of the obvious extension of F to all of $\mathbb{R} \times \mathbb{R}^d$. With the help of a slight abuse of notation, $\omega(F) = \{F\}$. ♦

Next, we consider asymptotically periodic functions.

Example 2.2.17. Suppose that for all compact subsets K of $\mathbb{R}^d$, the function $F : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is bounded and uniformly continuous on $[0, \infty) \times K$. Suppose also that there is a function $G : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ such that for some $p > 0$, $\tau_p G = G$ (so that G is p-periodic in t) and

$$\lim_{t\to\infty} \left|F(t, x) - G(t, x)\right| = 0, \quad \forall x \in \mathbb{R}^d. \tag{2.2.47}$$

Then $\omega(F) = \{\tau_\sigma G \colon 0 \le \sigma < p\}$.

Justification: Let $H \in \omega(F)$, and take a sequence (σ_n) such that $\sigma_n \to \infty$ and

$$\lim_{n\to\infty} F(t + \sigma_n, x) = H(t, x), \quad \forall t \in \mathbb{R}, \forall x \in \mathbb{R}^d. \tag{2.2.48}$$

(We will only use pointwise convergence here.) Fix $x \in \mathbb{R}^d$. Write each σ_n in the unique form

$$\sigma_n = a_n p + r_n, \tag{2.2.49}$$

where $r_n \in [0, p)$ and $a_n \in \mathbb{Z}$. By passing to a subsequence if necessary, it is no loss to assume that $r_n \to r$ for some $r \in [0, p]$. Now we use the triangle inequality to obtain

$$\begin{aligned}\left|F(t + \sigma_n, x) - G(t + r, x)\right| &= \left|F(t + a_n p + r_n, x) - G(t + a_n p + r, x)\right| &(2.2.50)\\ &\le \left|F(t + a_n p + r_n, x) - F(t + a_n p + r, x)\right| \\ &\quad + \left|F(t + a_n p + r, x) - G(t + a_n p + r, x)\right|. &(2.2.51)\end{aligned}$$

Because of the uniform continuity of F on $[0, \infty) \times \{x\}$,

$$\lim_{n\to\infty} \bigl|F(t + a_n p + r_n,\, x) - F(t + a_n p + r,\, x)\bigr| = 0. \tag{2.2.52}$$

Because of(2.2.47),

$$\lim_{n\to\infty} \bigl|F(t + a_n p + r,\, x) - G(t + a_n p + r,\, x)\bigr| = 0. \tag{2.2.53}$$

Altogether,

$$\lim_{n\to\infty} F(t + \sigma_n,\, x) = G(t + r,\, x) = \tau_r G(t,\, x). \tag{2.2.54}$$

If $r = p$, use $\tau_0 G = \tau_p G$ to conclude that the inclusion $\omega(F) \subset \{\tau_\sigma G\colon 0 \le \sigma < p\}$ holds. For the opposite inclusion, take $\sigma_n = a_n p + \sigma$, where a_n is a sequence in $\mathbb{N}$ such that the limit

$$H := \operatorname{co-lim}_{n\to\infty} \tau_{a_n p + \sigma} F \tag{2.2.55}$$

exists. (Such a choice of (a_n) is possible by application of Lemma 2.2.5 to the sequence $(np + \sigma)$.) Hence assumption (2.2.47) implies that $H = \tau_\sigma G$, because

$$\tau_\sigma G(t,\, x) = G(t + \sigma,\, x) = G(t + a_n p + \sigma,\, x). \tag{2.2.56}$$

This time a special case covers the t-periodic functions.

Example 2.2.18. If $F = F(t,\, x)$ is p-periodic in t, then $\omega(F)$ consists only of the obvious extensions of translates of F to all of $\mathbb{R} \times \mathbb{R}^d$. With the help of a slight abuse of notation, $\omega(F) = \{\tau_\sigma F\colon 0 \le \sigma < p\}$. ♦

We turn to one final class of functions. This will generalize the situation of Example 2.2.15. It is convenient in practice that in the situation of Example 2.2.15, the set $\omega(F)$ consists of only autonomous functions, indeed only one such function. We examine the possibility that $\omega(F)$ be a family of autonomous functions:

Definition 2.2.19. Suppose that for all compact subsets K of $\mathbb{R}^d$, the function $F : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is bounded and uniformly continuous on $[0, \infty) \times K$. We say that F is *asymptotically autonomous* if $\omega(F)$ consists entirely of autonomous functions $G(t, x) = G(x)$. ♦

We shall see that this includes not just functions with a pointwise autonomous limit as $t \to \infty$, but also functions that otherwise "level off" (such as smooth functions F that satisfy $D_t F(t, x) \to 0$ as $t \to \infty$). Here is an alternate characterization of asymptotic autonomy:

Theorem 2.2.20. *Suppose that for all compact subsets K of $\mathbb{R}^d$, the function $F : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is bounded and uniformly continuous on $[0, \infty) \times K$. Then the following conditions are equivalent.*

1. *F is asymptotically autonomous.*
2. *For every $x \in \mathbb{R}^d$, for all $\epsilon > 0$, and for all $r > 0$ there exists $T > 0$ such that $\left|F(s, x) - F(s + t, x)\right| < \epsilon$ provided that $s > T$ and $0 < t < r$.*

Proof. Assume that F is asymptotically constant. For contradiction, assume that there are $x_0 \in \mathbb{R}^d$, $\epsilon_0 > 0$, and $r_0 > 0$ and sequences $\sigma_n \to \infty$ and $(t_n) \subset (0, r_0)$ such that

$$\left|F(\sigma_n, x_0) - F(\sigma_n + t_n, x_0)\right| > \epsilon_0. \tag{2.2.57}$$

According to Lemma 2.2.5, we may pass to a subsequence, again denoted (σ_n) such that

$$G = \operatorname*{CO\text{-}lim}_{n\to\infty} \tau_{\sigma_n} F \tag{2.2.58}$$

exists in $\omega(F)$. In particular, G is autonomous and $F(\sigma_n + t, x) \to G(x)$ uniformly in $t \in [0, r]$ as $n \to \infty$. This stands in contradiction with (2.2.57).

For the converse, assume that 2 holds. Let $G \in \omega(F)$, and let (σ_n) be a corresponding sequence in $[0, \infty)$ such that $\sigma_n \to \infty$ and

$$G = \operatorname*{CO\text{-}lim}_{n\to\infty} \tau_{\sigma_n} F. \tag{2.2.59}$$

For contradiction, assume that G is not autonomous. Thus there are $x_0 \in \mathbb{R}^d$, $\epsilon_0 > 0$, and $s_0, t_0 \in \mathbb{R}$ such that $\left|G(s_0, x_0) - G(t_0, x_0)\right| = \epsilon_0$. According to 2, taking $r > \left|s_0 - t_0\right|$, we obtain $T > 0$ such that

$$\left|F(s_0 + \sigma_n, x_0) - F(t_0 + \sigma_n, x_0)\right| < \epsilon_0/2 \tag{2.2.60}$$

for all n sufficiently large that both $s_0 + \sigma_n$ and $t_0 + \sigma_n$ are larger than T.

Since $\left|F(s_0 + \sigma_n, x_0) - F(t_0 + \sigma_n, x_0)\right|$ converges to $\left|G(s_0, x_0) - G(t_0, x_0)\right| = \epsilon_0$ as n tends to ∞, this is a contradiction. □

Now we consider the case that $F(t, x)$ is of the "quasilinear" form $F(t, x) = A(t)p(x)$. As we shall see, it will be helpful to introduce the following notation, which is purposefully similar to the notation $\omega(F)$ already in place.

Definition 2.2.21. Let $A = A(t)$ be a bounded, uniformly continuous $\mathbb{R}^{d\times d}$-valued function on $[0, \infty)$. We define $\omega(A)$ to be the set of all $B : \mathbb{R} \to \mathbb{R}^{d\times d}$ such that for some sequence (σ_n) in $[0, \infty)$ with $\lim_{n\to\infty} \sigma_n = \infty$,

$$B = \operatorname*{CO\text{-}lim}_{n\to\infty} \tau_{\sigma_n} A. \tag{2.2.61}$$

That is, $A(\cdot + \sigma_n) \to B(\cdot)$ uniformly on compact sets. ♦

Lemma 2.2.22. *Let $A = A(t)$ be a bounded, uniformly continuous $\mathbb{R}^{d\times d}$-valued function on $[0, \infty)$. Let (σ_n) be a sequence in $[0, \infty)$ such that $\sigma_n \to \infty$. Then there is $B \in \omega(A)$ and a subsequence (σ_{n_k}) such that*

$$B = \operatorname*{co\text{-}lim}_{k\to\infty} \tau_{\sigma_{n_k}} A. \tag{2.2.62}$$

Proof. We apply Lemma 2.2.5 with $F(t, x) = A(t)x$. Thus, there is a subsequence (σ_{n_k}) such that $\left(\tau_{\sigma_{n_k}} F\right)$ converges uniformly on compact sets to some $G(t, x)$. It follows that for all $t, \lambda \in \mathbb{R}$ and $x, x' \in \mathbb{R}^d$,

$$G(t, \lambda x + x') = \lim_{k\to\infty} A(t + \sigma_{n_k})(\lambda x + x') \tag{2.2.63}$$

$$= \lambda \lim_{k\to\infty} A(t + \sigma_{n_k})x + \lim_{k\to\infty} A(t + \sigma_{n_k})x' \tag{2.2.64}$$

$$= \lambda G(t, x) + G(t, x'). \tag{2.2.65}$$

Hence, for each $t \in \mathbb{R}$, the partial map $G(t, \cdot)$ is a linear map, represented by a matrix $B(t)$. As a result, for each compact $J \times K \subset \mathbb{R} \times \mathbb{R}^d$, we have that

$$\lim_{k\to\infty} A(t + \sigma_{n_k})x = B(t)x \tag{2.2.66}$$

uniformly in $(t, x) \in J \times K$. By taking K to be the closed unit ball in $\mathbb{R}^d$, it follows that for all compact $J \subset \mathbb{R}$,

$$\lim_{k\to\infty} \left|A(t + \sigma_{n_k}) - B(t)\right| = 0 \tag{2.2.67}$$

uniformly in $t \in J$. This completes the proof. □

Lemma 2.2.23. *Let $A : [0, \infty) \to \mathbb{R}^{d\times d}$ be a matrix-valued function, and let $p : \mathbb{R}^d \to \mathbb{R}^d$. Define $F : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ by*

$$F(t,\, x) := A(t)p(x). \tag{2.2.68}$$

Assume that p is continuous, and that the function A is bounded and uniformly continuous. Then for all compact subsets K of $\mathbb{R}^d$, the function $F : [0,\, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is bounded and uniformly continuous on $[0,\, \infty) \times K$.

Moreover,

$$\omega(F) = \left\{Bp = B(t)p(x) \colon B \in \omega(A)\right\}. \tag{2.2.69}$$

Proof. Let $M > 0$ be a bound for the matrix norm $\left|A(t)\right|$ for all $t \geq 0$. Let K be any compact subset of $\mathbb{R}^d$. For all $t \geq 0$ and all $x \in K$,

$$\left|A(t)p(x)\right| \leq M \left|p(x)\right|. \tag{2.2.70}$$

Since p is bounded on the compact set K, this shows that F is bounded on $[0,\, \infty) \times K$. Now let $\epsilon > 0$, and let $\delta > 0$ be such that $\left|A(t) - A(s)\right| < \epsilon$ whenever $\left|t - s\right| < \delta$. Then, for all $x,\, y \in K$, we have

$$\begin{aligned} \left|F(t,\, x) - F(s,\, y)\right| &\leq \left|\big(A(t) - A(s)\big)p(x)\right| + \left|A(s)p(x-y)\right| && (2.2.71)\\ &\leq \epsilon \left|p(x)\right| + M \left|p(x) - p(y)\right|. && (2.2.72)\end{aligned}$$

by the triangle quantity. By use of the boundedness and uniform continuity of the continuous function p on the compact set K, we can refine the choice of δ so that $\left|F(t,\, x) - F(s,\, y)\right| \leq \epsilon$ whenever $\left|t - s\right| < \delta$ and $x,\, y \in K$, $\left|x - y\right| < \delta$. This shows the uniform continuity of F on $[0,\, \infty) \times K$.

Now choose $G \in \omega(F)$, and let (σ_n) be a corresponding sequence in $[0, \infty)$ such that

$$G = \underset{n\to\infty}{\text{co-lim}}\, \tau_{\sigma_n} F. \tag{2.2.73}$$

According to Lemma 2.2.22, there is a subsequence (σ_{n_k}) and $B \in \omega(A)$ such that

$$B = \underset{k\to\infty}{\text{co-lim}}\, \tau_{\sigma_{n_k}} A. \tag{2.2.74}$$

Hence, for all $(t, x) \in \mathbb{R} \times \mathbb{R}^d$,

$$\begin{aligned} G(t, x) &= \lim_{k\to\infty} F(t + \sigma_{n_k}, x) && (2.2.75)\\ &= \lim_{k\to\infty} A(t + \sigma_{n_k}) p(x) && (2.2.76)\\ &= B(t)p(x), && (2.2.77) \end{aligned}$$

which proves the inclusion

$$\omega(F) = \subset Bp = B(t)p(x) B \in \omega(A). \tag{2.2.78}$$

The reverse inclusion is proved, similarly, using Lemma 2.2.5 in place of Lemma 2.2.22. $\square$

The spirit of the preceding lemma may be captured by the slogan "$\omega(F) = \omega(A)$", but of course this equation cannot be taken literally. Lemma 2.2.23 allows some earlier results to be rephrased as calculations of $\omega(A)$ when F is of the form $F(t, x) = A(t)p(x)$.

Example 2.2.24. Suppose that $A = A(t)$ is a bounded, uniformly continuous $\mathbb{R}^{d\times d}$-valued function on $[0, \infty)$. Then:

1. If $A_0 = \lim_{t\to\infty} A(t)$ exists, then $\omega(A) = \{A_0\}$.

2. If there is a p-periodic matrix function B on $\mathbb{R}$ such that $\lim_{t\to\infty} \left|A(t) - B(t)\right| = 0$, then

$$\omega(A) = \{\tau_\sigma B \colon 0 \le \sigma < p\}\,. \tag{2.2.79}$$

3. If for all $\epsilon > 0$ and all $r > 0$ there exists $T > 0$ such that $\left|A(s) - A(s+t)\right| < \epsilon$ whenever $s > T$ and $0 < t < r$, then $\omega(A)$ consists entirely of constant functions, and we say that A is *asymptotically autonomous* or *asymptotically constant*. ♦

Justification. The three statements follow from Example 2.2.15, Example 2.2.17, and Theorem 2.2.20, respectively, via Lemma 2.2.23. □

Remark 2.2.25. Notice that A is asymptotically constant when $A_0 = \lim_{t\to\infty} A(t)$ exists, and also when A is differentiable with $\lim_{t\to\infty} \dot{A}(t) = 0$. Also, when A is asymptotically constant one has

$$\omega(A) = \bigcap_{n\in\mathbb{N}} \overline{A([n,\,\infty))}, \tag{2.2.80}$$

which is verified as follows. If $B \in \omega(A)$, then there is a sequence $\sigma_n \to \infty$ such that $A(0+\sigma_n) \to B$ as $n \to \infty$. Since Each set $A([k,\,\infty))$ contains a tail of the sequence $\big(A(\sigma_n)\big)$, this proves one inclusion. For the reverse inclusion, suppose that

$$B \in \bigcap_{n\in\mathbb{N}} \overline{A([n,\,\infty))}. \tag{2.2.81}$$

Then, for each $n \in \mathbb{N}$, there is a sequence $\big(\sigma_k^{(n)}\big) \in [n,\,\infty)$ such that $A(\sigma_k^{(n)}) \to B$ as $k \to \infty$. For each $N \in \mathbb{N}$ define a new sequence (σ_n) by setting $\sigma_n = \sigma_k^{(n)}$, where k is chosen sufficiently large that $\left|A\left(\sigma_k^{(n)}\right) - B\right| < (1/n)$. Thus, $\sigma_n \to \infty$ and $A(\sigma_n) \to B$ as $n \to \infty$. According to Lemma 2.2.22, we can pass to a subsequence $\big(\sigma_{n_k}\big)$ such that for some $H \in \omega(A)$,

$$H = \operatorname*{CO-lim}_{k\to\infty} \tau_{\sigma_{n_k}} A. \tag{2.2.82}$$

But H is constant because A is assumed to be asymptotically constant. Since $H(0) = B$, $H = B$ and therefore $B \in \omega(A)$. $\Diamond$

For later use, we use Lemma 2.2.23 to record a special case of Theorem 2.2.14.

Theorem 2.2.26. *Let P be any projection on $\mathbb{R}^d$. Suppose that $A = A(t)$ is a bounded, uniformly continuous $\mathbb{R}^{d\times d}$-valued function on $[0, \infty)$, and put $F(t, x) = A(t)x$. Also suppose that for no $B \in \omega(A)$ does $\dot{u}(t) + B(t)u(t) = 0$ have a nonzero solution $u \in C_b^1(\mathbb{R})$.*

Then the linear operator $\Phi_{F,P} : C^1_{\{0\}}\big([0, \infty), \mathbb{R}^d\big) \to C_{\{0\}}\big([0, \infty), \mathbb{R}^d\big) \times \operatorname{rge} P$ is proper on the closed bounded subsets of $C^1_{\{0\}}\big([0, \infty), \mathbb{R}^d\big)$.

Proof. This is essentially a corollary of Theorem 2.2.14. Notice that $\omega_0(A) = \{0\}$, because any element of $\omega_0(A)$ is a constant solution to one of the equations $\dot{u}(t)+B(t)u(t) = 0$. Thus, $\omega_0(A)$ is totally disconnected. (We remark that since $\omega_0(A)$ is a linear subspace of $\mathbb{R}^d$, we could not allow any constant nonzero solutions to any of the equations $\dot{u}(t) + B(t)u(t) = 0$ without destroying the total disconnectedness of $\omega_0(A)$. This is why we prohibit "nonzero" solutions instead of just "non-constant" solutions.) $\square$

2.3 THE FREDHOLM PROPERTY AND INDEX

2.3.1 Reduction of the Fredholm property to the linear setting

As seen in Chapter 1, one ingredient of the degree argument is to show that the underlying map of Banach spaces—in this case $\Phi_{F,P} : C^1_{\{0\}} \to C_{\{0\}} \times \operatorname{rge} P$—is Fredholm of index zero. We develop several ways to check this. Each method involves analysis of the asymptotic behavior of solutions to certain linear homogeneous equations $\dot{u}(t)+A(t)u(t) = 0$. Of primary importance for each method is the ability to check that there are no bounded solutions that do not tend to zero as $t \to \infty$. One also needs to know at least the dimension of the space of solutions that do tend to zero as $t \to \infty$. In the case of a constant $A(t) = A_0$, one just checks that A_0 has no imaginary eigenvalues, and then counts the eigenvalues with positive real part. The periodic case is also fairly simple. As we shall see, more sophisticated analysis is needed in a more general setting, and the set $\omega(A)$ can play a key role.

We invite the reader to turn to Section 1.4 to review Definitions 1.4.1 and 1.4.3 and Properties 1.4.4 and 1.4.5. These properties are the basis of the following lemma, which reduces the Fredholm property and index of $\Phi_{F,\,P}$ to that of $D\Phi_{F,\,P}(0)$.

Lemma 2.3.1. *Let P be any projection on $\mathbb{R}^d$. Suppose that $F : [0,\,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies* (2.1.1a)*,* (2.1.1b)*, and* (2.1.1c) *on page 16. Suppose also that the linear map $D\Phi_{F,\,P}(0)$ is Fredholm of index zero,* i.e. *$D\Phi_{F,\,P}(0)$ satisfies the following condition:*[7]

$$\dim\ker D\Phi_{F,\,P}(0) = \operatorname{codim}\operatorname{rge} D\Phi_{F,\,P}(0) < \infty, \tag{2.3.1}$$

where the co-dimension is with respect to the ambient space $C_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big) \times \operatorname{rge} P$. Then $\Phi_{F,\,P}$ is Fredholm of index zero from $C^1_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big)$ into $C_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big) \times \operatorname{rge} P$.

Proof. According to the assumptions on F, Corollaries 2.1.3 and 2.1.8 ensure that for all $u \in C^1_{\{0\}}$, $D\Phi(u)$ is a well-defined bounded linear operator from $C^1_{\{0\}}$ into $C_{\{0\}} \times \operatorname{rge} P$ and is moreover a compact perturbation of $D\Phi_{F,\,P}(0)$. From this, the Lemma follows directly from the invariance of the Fredholm property and index under compact perturbations (see Property 1.4.5 on page 10). □

[7]Of course, this condition is satisfied in the special case that $D\Phi_{F,\,P}(0)$ is an isomorphism of $C^1_{\{0\}}$ onto $C_{\{0\}} \times \operatorname{rge} P$.

Remark 2.3.2. In the above lemma, it is clear that $D\Phi_{F,\,P}(0)$ can be replaced by $D\Phi_{F,\,P}(u)$, for any $u \in C^1_{\{0\}}$. Of course, $u = 0$ will almost always be the most convenient choice. $\Diamond$

2.3.2 Isomorphisms

According to Lemma 2.3.1, one way to guarantee that $\Phi_{F,\,P}$ is Fredholm of index zero is to know that $D\Phi_{F,\,P}(0)$ is an isomorphism. This condition, while not necessary, is nevertheless practical to verify in certain cases. Also, the knowledge that $D\Phi_{F,\,P}(0)$ is an isomorphism can help with a different part of the degree argument; see Section 2.3.6. Here are two cases when we can verify this. We do not verify these examples now. Instead we refer to later examples for details. Put $A(t) := D_xF(t,\,0)$, so that $D\Phi_{F,\,P}(0)$ maps $u = u(t)$ to $\Lambda_{A,\,P}u = \Lambda_{A,\,P}u(t) := \big(\dot{u}(t) + A(t)u(t),\, Pu(0)\big)$.

Example 2.3.3. Suppose that $A(t) = A_0$ is constant. Suppose that A_0 has no pure imaginary eigenvalues, and that the algebraic number of those eigenvalues of A_0 with positive real part is equal to the rank of P. Suppose also that the kernel of P contains no nontrivial generalized eigenvector of A_0 that corresponds to an eigenvalue with positive real part. For example, if the range of P happens to be the sum of the generalized eigenspaces of A corresponding to eigenvalues with positive real part (and if A_0 has no imaginary eigenvalues), then these conditions are met. In this case, $\Lambda_{A,\,P}$ is an isomorphism. See Example 2.3.24.

Also, with use of the Floquet theory, similar results hold in case A is periodic. See Example 2.3.25 for details. $\blacklozenge$

Remark 2.3.4. In general, it is much harder to tell that $D\Phi_{F,\,P}(0)$ is an isomorphism than it is to tell that it is Fredholm of index zero.[8] Rabier and the author explore this issue further

[8]However, Corollaries 2.3.23 and 2.3.17 imply that these properties are equivalent when $P = I$ or $P = 0$.

in [MR], where conditions are found for $\Lambda_{A,\,P}$ to be an isomorphism, even when $A(t)$ is not bounded in t. $\Diamond$

2.3.3 Exponential dichotomies

It is possible in certain cases to verify that $D\Phi_{F,\,P}(0)$ is Fredholm of index zero even if $D\Phi_{F,\,P}(0)$ is not an isomorphism. For this purpose we take a short detour into the stability theory of linear dynamical systems in order to study the concept of exponential dichotomies. This will prove worthwhile for the following three reasons. First, there is a simply stated condition involving exponential dichotomies that is necessary and sufficient for $\Phi_{F,\,P}$ to be Fredholm of index zero. Second, there are several simple conditions in the literature that are sufficient to ensure the exponential dichotomy property. Third, even though a different part of the degree argument is helped by the assumption that $D\Phi_{F,\,P}(0)$ is an isomorphism, this assumption is not necessary for even that purpose. We will return to this issue in Section 2.3.6.

Having made this brief attempt at motivation, let us now discuss exponential dichotomies. Let $A = A(t) \in \mathbb{R}^{d\times d}$ denote a continuous, matrix valued function on $[0,\,\infty)$. We consider solutions of the homogeneous linear differential equation

$$\dot{u}(t) + A(t)u(t) = 0 \tag{2.3.2}$$

and also of the associated inhomogeneous equation

$$\dot{u}(t) + A(t)u(t) = f(t), \tag{2.3.3}$$

where $u : [0,\,\infty) \to \mathbb{R}^d$ is the unknown. By D_A we shall denote the operator

$$D_A u(t) := \dot{u} + A(t)u(t) \tag{2.3.4}$$

suggested by the left hand side of equations (2.3.2) and (2.3.3). We view D_A as an operator acting from $C^1_{\{0\}}\big([0, \infty), \mathbb{R}^d\big)$ into $C_{\{0\}}\big([0, \infty), \mathbb{R}^d\big)$. Because A is bounded in t, this is justified by Corollary 2.1.3. Indeed, that theorem implies that D_A is continuous from $C^1_{\{0\}}$ into $C_{\{0\}}$. We denote by $U = U(t)$ the fundamental matrix solution to the matrix problem

$$\dot{U}(t) = -A(t)U(t) \tag{2.3.5}$$

$$U(0) = I. \tag{2.3.6}$$

Recall the following property of the fundamental matrix solution U (see Hartman [Har64]), which will be used often throughout this section:

Lemma 2.3.5. *For any $\xi \in \mathbb{R}^d$, the function u_ξ defined by $u_\xi(t) = U(t)\xi$ is a solution of (2.3.2). Conversely, every solution u of (2.3.2) satisfies $u(t) = U(t)u(0)$.*

Now we provide the standard definitions for the two kinds of dichotomy (exponential and ordinary), and provide for later use some simple necessary and sufficient conditions. The concepts and results presented in Section 2.3.3 are not new; see for example [Har64], [Cop78], [MS66], and [SS78]. For clarity and completeness, we choose to present just that portion of the theory which we shall use in the sequel.

Let Π be a linear projection defined on $\mathbb{R}^d$. Let $T \geq 0$. The equation (2.3.2) is said to admit a dichotomy on $[T, \infty)$ with associated projection Π if there exist constants $K > 0$ and $\alpha \geq 0$ such that for all $s, t \geq 0$,

$$\big\|U(t)\Pi U(s)^{-1}\big\| \leq Ke^{\alpha(s-t)}, \quad T \leq s \leq t, \tag{2.3.7}$$

and

$$\big\|U(t)(I - \Pi)U(s)^{-1}\big\| \leq Ke^{\alpha(t-s)}, \quad T \leq t \leq s. \tag{2.3.8}$$

The equation (2.3.2) is said to admit an ordinary dichotomy if the above definition holds with $\alpha = 0$, and is said to admit an exponential dichotomy if α may be taken to be positive. Trivially, an ordinary dichotomy is admitted whenever an exponential dichotomy is admitted.

Remark 2.3.6. It is easy to see that if A admits an exponential dichotomy on $[T, \infty)$ with associated projection Π, then A also admits an exponential dichotomy on $[0, \infty)$ with the same projection Π. Indeed, since the fundamental matrix solution $U(t)$ is continuous and invertible, there exists

$$M := \max\left\{\max_{0\le t\le T}\left|U(t)\right|,\ \max_{0\le t\le T}\left|U(t)^{-1}\right|\right\} < \infty. \tag{2.3.9}$$

If $0 \le s \le t \le T$, then

$$\left|U(t)\Pi U(s)^{-1}\right| \le M^2. \tag{2.3.10}$$

If $0 \le s \le T \le t$, then

$$\begin{aligned} \left|U(t)\Pi U(s)^{-1}\right| &= \left|U(t)\Pi U(T)^{-1}U(T)U(s)^{-1}\right| &(2.3.11)\\ &\le M^2\left|U(t)\Pi U(T)^{-1}\right|. &(2.3.12)\end{aligned}$$

These inequalities show that after increasing K if necessary, inequality (2.3.7) holds with T replaced by 0. A similar argument takes care of inequality (2.3.8) with T replaced by 0.

This remark can be helpful when A has a particular property for sufficiently large T. For example, in the next two examples, it would be enough to assume that $A(t)$ is eventually constant, or eventually periodic. $\Diamond$

Example 2.3.7. Let $A = A_0$ be constant, so that $U(t) = \exp(-tA_0) := \sum_{k=0}^{\infty}(-tA_0)^k/k!$. Let Π be the spectral projection of $\mathbb{R}^d$ onto the sum of the generalized eigenspaces of A_0 which correspond to eigenvalues with positive real part. Then A_0 admits an exponential dichotomy with associated projection Π. Note that the rank of Π is equal to the algebraic number of eigenvalues of A_0 that have positive real part.

Conversely, if A_0 has any pure imaginary eigenvalues, then A_0 does not admit an exponential dichotomy. This is a standard and model example of exponential dichotomy. See Coppel [Cop78]. ♦

Example 2.3.8. The case that $A(t+p) = A(t)$ is p-periodic is reduced to the constant case as follows. According to the Floquet Theory (see Hsieh and Sibuya [HS99], pages 87-89), by setting

$$B = \frac{1}{2p}\log\bigl(U(p)^2\bigr) \tag{2.3.13}$$

and

$$Q(t) = U(t)\exp(tB), \tag{2.3.14}$$

the problem $\dot{u}(t) + A(t)u(t) = 0$ is transformed to

$$\dot{v}(t) + Bv(t) = 0 \tag{2.3.15}$$

for $u(t) = Q(t)v(t)$. Notice that $Q(0) = I$. Hence an initial condition $Pu(0) = \xi$ becomes simply $Pv(0) = \xi$.

Let $V = V(t)$ be the fundamental matrix solution for the transformed problem, so that $U(t) = Q(t)V(t)$. Since the change of variables $Q(t)$ is invertible, continuous and p-periodic, the inequalities

$$\bigl\|U(t)\Pi U(s)^{-1}\bigr\| \le Ke^{\alpha(s-t)}, \quad s \le t, \tag{2.3.16}$$

and

$$\left\|U(t)(I-\Pi)U(s)^{-1}\right\| \leq Ke^{\alpha(t-s)}, \quad t \leq s \tag{2.3.17}$$

from the definition of exponential dichotomy are transformed to

$$\left\|V(t)\Pi V(s)^{-1}\right\| \leq K' e^{\alpha(s-t)}, \quad s \leq t \tag{2.3.18}$$

and

$$\left\|V(t)(I-\Pi)V(s)^{-1}\right\| \leq K' e^{\alpha(t-s)}, \quad t \leq s, \tag{2.3.19}$$

where the choice of K' depends on bounds for $\left|Q(t)\right|$ and $\left|Q(t)^{-1}\right|$. These bounds exist by invertibility, continuity, and periodicity.

Hence, A admits an exponential dichotomy with projection Π if and only if B does the same. Also note that each eigenvalue of B is obtained as twice the logarithm of the modulus of a corresponding eigenvalue of $U(p)$, and is associated to the same generalized eigenspace. (See the calculation of $\log\big(U(p)^2\big)$ on page 88 of [HS99].) Hence, we have that the equation

$$\dot{u}(t) + A(t)u(t) = 0 \tag{2.3.20}$$

admits an exponential dichotomy on $[0, \infty)$ if and only if none of the eigenvalues of $U(p)$ have modulus 1. In this case, one may take Π to be the spectral projection of $\mathbb{R}^d$ onto the generalized eigenspace of $U(p)$ corresponding to eigenvalues of modulus greater than 1. Also, the rank of Π is equal to the algebraic number of eigenvalues of $U(p)$ that have modulus greater than 1. ♦

An exponential dichotomy describes a splitting of $\mathbb{R}^d$ into a stable subspace $\operatorname{rge}\Pi$ and an unstable subspace $\ker\Pi$. Solutions that take an initial value in the stable subspace will exhibit exponential decay in the forward direction, while those starting in $\ker\Pi$ decay exponentially fast in the reverse direction. Quite naturally, this behavior entails corresponding exponential growth in the opposite directions. More precisely:

Lemma 2.3.9. *Let Π be a linear projection defined on $\mathbb{R}^d$, and let $A = A(t) \in \mathbb{R}^{d\times d}$ be a continuous, matrix valued function on $[0, \infty)$. Then the following are equivalent:*

1. *The equation (2.3.2) admits an exponential dichotomy over $[0, \infty)$ with associated projection Π.*
2. *There exist M, $K > 0$ and $\alpha > 0$ such that for all $\xi \in \mathbb{R}^d$ and all s, $t \geq 0$,*

$$\left|U(t)\Pi U(s)^{-1}\xi\right| \geq Me^{\alpha(s-t)}\left|U(s)\Pi U(s)^{-1}\xi\right|, \quad t \leq s, \tag{2.3.21}$$

$$\left|U(t)(I-\Pi)U(s)^{-1}\xi\right| \geq Me^{\alpha(t-s)}\left|U(s)(I-\Pi)U(s)^{-1}\xi\right|, \quad s \leq t, \tag{2.3.22}$$

and

$$\left|U(s)\Pi U(s)^{-1}\right| \leq K. \tag{2.3.23}$$

Proof. First, assume that equation (2.3.2) admits an exponential dichotomy over $[0, \infty)$ with associated projection Π. Let $\xi \in \mathbb{R}^d$, and let $0 \leq t \leq s$. Noting that

$$U(s)\Pi U(s)^{-1} = U(s)\Pi U(t)^{-1}U(t)\Pi U(s)^{-1}, \tag{2.3.24}$$

we have

$$\left|U(s)\Pi U(s)^{-1}\xi\right| = \left|U(s)\Pi U(t)^{-1}\big(U(t)\Pi U(s)^{-1}\xi\big)\right| \tag{2.3.25}$$

$$\leq Ke^{\alpha(t-s)}\left|U(t)\Pi U(s)^{-1}\xi\right|, \tag{2.3.26}$$

where we have made use of inequality (2.3.7) from the definition of exponential dichotomy, after interchanging the roles of s and t in (2.3.7). Hence,

$$\left|U(t)\Pi U(s)^{-1}\xi\right| \geq K^{-1}e^{\alpha(s-t)}\left|U(s)\Pi U(s)^{-1}\xi\right|, \tag{2.3.27}$$

which establishes (2.3.21) for $M = K^{-1}$. Similarly, (2.3.22) follows from

$$U(s)(I-\Pi)U(s)^{-1} = U(s)(I-\Pi)U(t)^{-1}U(t)(I-\Pi)U(s)^{-1} \tag{2.3.28}$$

with use of (2.3.8). Finally, (2.3.23) is just (2.3.7) in the special case that $s = t$.

For the converse, suppose that $0 \leq s \leq t$, and let $\xi \in \mathbb{R}^d$. We deduce from (2.3.21) and equation (2.3.24) that

$$\left|U(s)\Pi U(s)^{-1}\xi\right| \geq Me^{\alpha(s-t)}\left|U(t)\Pi U(s)^{-1}\xi\right|. \tag{2.3.29}$$

Because of (2.3.23), this implies that

$$C\left|\xi\right| \geq Me^{\alpha(s-t)}\left|U(t)\Pi U(s)^{-1}\xi\right|. \tag{2.3.30}$$

Hence,

$$\left|U(t)\Pi U(s)^{-1}\xi\right| \leq CM^{-1}e^{\alpha t-s}\left|\xi\right| \tag{2.3.31}$$

for all $\xi \in \mathbb{R}$, which establishes (2.3.7) in the definition of exponential dichotomy. Similarly, when $0 \leq t \leq s$, we deduce from (2.3.22) that

$$\left|U(s)(I-\Pi)U(s)^{-1}\xi\right| \geq Me^{\alpha(s-t)}\left|U(t)(I-\Pi)U(s)^{-1}\xi\right|. \tag{2.3.32}$$

Since

$$\|U(s)(I-\Pi)U(s)^{-1}\| = \|I - U(s)\Pi U(s)^{-1}\| \leq 1 + C, \tag{2.3.33}$$

this implies that

$$\left|U(t)(I-\Pi)U(s)^{-1}\xi\right| \leq (1+C)M^{-1}e^{\alpha(t-s)}\left|\xi\right|, \tag{2.3.34}$$

proving that (2.3.8) holds in the definition of exponential dichotomy. □

Remark 2.3.10. The growth conditions (2.3.22) and (2.3.21) do not alone imply the admission of an exponential dichotomy, as is seen by an example in chapter 2 of [Cop78]. The third condition (2.3.23) may be interpreted as the existence of a positive uniform lower bound for the angle between the subspaces $U(t)\operatorname{rge}\Pi$ and $U(t)\ker\Pi$, which may be thought of respectively as the trajectories of the stable and unstable subspaces at time t. For later convenience, we record the following trivial corollary of Lemma 2.3.9. ◊

Corollary 2.3.11. *Suppose that the equation (2.3.2) admits an exponential dichotomy over* $[0, \infty)$ *with associated projection* Π. *Then there exist positive constants* K *and* α *such that for any* $\xi \in \mathbb{R}^d$ *and* $t \geq 0$*:*

$$\left|U(t)\Pi\xi\right| \leq Ke^{-\alpha t}\left|\xi\right|, \text{ and} \tag{2.3.35}$$

$$\left|U(t)(I-\Pi)\xi\right| \geq K^{-1}e^{\alpha t}\left|(I-\Pi)\xi\right| \tag{2.3.36}$$

Proof. The first estimate follows directly from (2.3.7), taking $s = 0$. The second follows from (2.3.22), again taking $s = 0$. □

The following is sometimes of help.[9] If it is known that A admits an exponential dichotomy, it is really only the range of the associated projection that is critical:

Lemma 2.3.12 (Coppel[Cop78]). *Suppose that* A *admits an exponential dichotomy on* $[0, \infty)$, *with associated projection* Π. *Let* Π' *be any projection on* $\mathbb{R}^d$ *with the same range as* Π. *Then* A *also admits an exponential dichotomy on* $[0, \infty)$ *with associated projection* Π'.

[9] Note that Lemma 2.3.12 is not true when working with exponential dichotomies on the whole line $\mathbb{R}$.

Remark 2.3.13. Note that in the above lemma, there is no claim that Π' satisfies the inequalities (2.3.7) and (2.3.8) for the same constants K and α; in general it will not. $\Diamond$

2.3.4 Exponential dichotomies and the Fredholm property

We now show the close relationship between exponential dichotomy and the Fredholm property. As we will show, the concepts are essentially the same. This was proved in a slightly different setting by Palmer [Pal84, Pal88]. Because Palmer works with spaces that do not incorporate decay at infinity, and for maximim clarity, we prove this result here. First, we prove that the former entails the latter.

Theorem 2.3.14. *Suppose that the bounded, continuous matrix valued function $A = A(t)$ admits an exponential dichotomy on $[0, \infty)$, with projection Π. Then the continuous linear operator*

$$D_A : C^1_{\{0\}}\big([0, \infty), \mathbb{R}^d\big) \to C_{\{0\}}\big([0, \infty), \mathbb{R}^d\big) \tag{2.3.37}$$

defined by the relation

$$D_A u(t) := \dot{u}(t) + A(t)u(t) \tag{2.3.38}$$

is surjective with $\dim \ker D_A = \dim \operatorname{rge} \Pi$. *In particular, D_A is Fredholm of index* $\dim \operatorname{rge} \Pi$.

Additionally, the kernel of D_A is characterized by

$$\ker D_A = \{u_\xi \mid \xi \in \operatorname{rge} \Pi\}, \tag{2.3.39}$$

where u_ξ is defined by

$$u_\xi(t) := U(t)\xi. \tag{2.3.40}$$

Proof. Next, the kernel of D_A consists of all functions in $C^1_{\{0\}}$ that satisfy the homogeneous equation (2.3.2). Recall that the set W of all such solutions is a vector space, and that the correspondence between a vector $\xi \in \mathbb{R}^d$ and the solution $u_\xi \in W$ given by

$$u_\xi(t) = U(t)\xi \tag{2.3.41}$$

is an isomorphism of vector spaces. Therefore, $\ker D_A = W \cap C^1_{\{0\}}$, and

$$\dim \ker D_A = \dim\{\xi \in \mathbb{R}^d \mid u_\xi \in C^1_{\{0\}}\}. \tag{2.3.42}$$

To find this dimension, we consider the effect of two possible choices of ξ. First, suppose that $\xi \in \operatorname{rge} \Pi$. Then

$$\begin{aligned} \left|u_\xi(t)\right| &= \left|U(t)\xi\right| && (2.3.43)\\ &= \left|U(t)\Pi\xi\right| && (2.3.44)\\ &\leq Ke^{-\alpha t}\left|\xi\right|, && (2.3.45)\end{aligned}$$

according to (2.3.35) in Corollary 2.3.11. Since A is assumed to be bounded and u satisfies (2.3.2), the derivative of u_ξ also exhibits exponential decay. Since A is continuous, these bounds ensure that $\dot{u}_\xi \in C_{\{0\}}$, and hence that $u_\xi \in C^1_{\{0\}}$. Thus,

$$\{u_\xi \mid \xi \in \operatorname{rge} \Pi\} \subset \ker D_A. \tag{2.3.46}$$

In case $\xi \notin \operatorname{rge} \Pi$, we have $(I - \Pi)\xi \neq 0$. We decompose the solution along Π, giving

$$\begin{aligned} \left|u_\xi(t)\right| &= \left|U(t)\Pi\xi + U(t)(I-\Pi)\xi\right| && (2.3.47)\\ &= \left|u_{\Pi\xi}(t) + u_{(I-\Pi)\xi}(t)\right|. && (2.3.48)\end{aligned}$$

We already know that $u_{\Pi\xi} \in C^1_{\{0\}}$. However, using (2.3.36) in Corollary 2.3.11,

$$\left|U(t)(I-\Pi)\xi\right| \geq K^{-1}e^{\alpha t}\left|(I-\Pi)\xi\right|. \tag{2.3.49}$$

Since $(I-\Pi)\xi$ is assumed to be nonzero, this shows that $u_{(I-\Pi)\xi}$ has exponential growth and so the sum $u_\xi = u_{\Pi\xi} + u_{(I-\Pi)\xi}$ cannot be in $C^1_{\{0\}}$. We have shown that

$$\ker D_A = \{u_\xi \mid \xi \in \operatorname{rge}\Pi\}, \tag{2.3.50}$$

which verifies (2.3.39) and hence also that

$$\dim\ker D_A = \dim\operatorname{rge}\Pi, \tag{2.3.51}$$

as advertised. To show that D_A is surjective, let $f \in C_{\{0\}}$ and note that $f \in \operatorname{rge} D_A$ if and only if there is some $u \in C^1_{\{0\}}$ that solves the inhomogeneous equation (2.3.3) for f. Recall that the set of all such solutions consists of all functions in $C^1_{\{0\}}$ of the form

$$u_{\xi,f}(t) = U(t)\left(\xi + \int_0^t (U(s))^{-1}f(s)\,\mathrm{d}s\right) \tag{2.3.52}$$

for some $\xi \in \mathbb{R}^d$. In other words, $f \in \operatorname{rge} D_A$ if and only if there is some choice of $\xi \in \mathbb{R}^d$ such that $u_{\xi,f} \in C^1_{\{0\}}$. We decompose $u_{\xi,f}$ along the projection Π as follows:

$$\begin{aligned} u_{\xi,f}(t) = U(t)\Pi\xi &+ \int_0^t U(t)\Pi(U(s))^{-1}f(s)\,\mathrm{d}s \\ &+ U(t)\left((I-\Pi)\xi + \int_0^t (I-\Pi)(U(s))^{-1}f(s)\,\mathrm{d}s\right). \end{aligned} \tag{2.3.53}$$

The first term on the right hand side is just $u_{\Pi\xi}(t)$, and we saw in (2.3.43) that $u_{\Pi\xi} \in C^1_{\{0\}}$ for every choice of ξ. In particular, we point out that

$$u_{\Pi\xi} \in C_{\{0\}}, \quad \forall\xi \in \mathbb{R}^d. \tag{2.3.54}$$

Using (2.3.35) in Corollary 2.3.11, we estimate the size of the second term on the right hand side of (2.3.53):

$$\left| \int_0^t U(t)\Pi(U(s))^{-1} f(s)\,\mathrm{d}s \right| \le \int_0^t K e^{\alpha(s-t)} \left|f(s)\right| \mathrm{d}s. \tag{2.3.55}$$

We now choose $\epsilon > 0$ and find some $T > 0$ so large that $\left|f(t)\right| < \epsilon$ for $t > T$. Then for $t > T$

$$\int_0^t K e^{\alpha(s-t)} \left|f(s)\right| \mathrm{d}s \le K \left\|f\right\|_\infty \int_0^T e^{\alpha(s-t)}\,\mathrm{d}s + K\epsilon \int_T^t e^{\alpha(s-t)}\,\mathrm{d}s \tag{2.3.56}$$

$$= \frac{K\left\|f\right\|_\infty}{\alpha}(e^{\alpha(T-t)} - e^{-\alpha t}) + \frac{K\epsilon}{\alpha}(1 - e^{\alpha(T-t)}). \tag{2.3.57}$$

This expression is bounded by $2K\epsilon/\alpha$ for sufficiently large t. Since $\epsilon > 0$ is arbitrary, we conclude that if $f \in C_{\{0\}}$, then $\int_0^t K e^{\alpha(s-t)} \left|f(s)\right| \mathrm{d}s$ tends to zero as $t \to \infty$. Hence we have shown that the second term on the right hand side of (2.3.53) defines an element of $C_{\{0\}}$, for any choice of $\xi \in \mathbb{R}^d$:

$$\left(t \mapsto \int_0^t U(t)\Pi(U(s))^{-1} f(s)\,\mathrm{d}s \right) \in C_{\{0\}}. \tag{2.3.58}$$

We move on to the third (final) term on the right side of (2.3.53). Notice that the integral

$$\int_0^\infty (I - \Pi)(U(s))^{-1} f(s)\,\mathrm{d}s \tag{2.3.59}$$

exists in the unstable space $\operatorname{rge}(I - \Pi)$; the convergence is because of the exponential decay in (2.3.8). We have not yet placed any restriction on ξ; now we require

$$(I - \Pi)\xi = -\int_0^\infty (I - \Pi)(U(s))^{-1} f(s)\,\mathrm{d}s. \tag{2.3.60}$$

That is, let ξ be any vector of the form

$$\xi = -\int_0^\infty (I - \Pi)(U(s))^{-1} f(s)\,\mathrm{d}s + \eta, \quad \eta \in \operatorname{rge}\Pi. \tag{2.3.61}$$

With this restriction on ξ, we can estimate the third term in the right side of (2.3.53) as follows, with use of (2.3.8):

$$\begin{aligned}
&\left|U(t)\left((I-\Pi)\xi + \int_0^t (I-\Pi)(U(s))^{-1}f(s)\,\mathrm{d}s\right)\right| \\
&\qquad = \left|\int_t^\infty U(t)(I-\Pi)(U(s))^{-1}f(s)\,\mathrm{d}s\right| && (2.3.62)\\
&\qquad \le \int_t^\infty Ke^{\alpha(t-s)}\left|f(s)\right|\mathrm{d}s && (2.3.63)\\
&\qquad \le K\alpha^{-1}\sup_{s\ge t}\left|f(s)\right|. && (2.3.64)
\end{aligned}$$

We may once again use the decay of f to conclude that the third and final term on the right side of (2.3.53) defines a function in $C_{\{0\}}$:

$$\left(t \mapsto U(t)\left((I-\Pi)\xi + \int_0^t (I-\Pi)(U(s))^{-1}f(s)\,\mathrm{d}s\right)\right) \in C_{\{0\}}. \tag{2.3.65}$$

In summary, we now know that under an appropriate choice (2.3.61) of ξ, the results (2.3.54), (2.3.58), and (2.3.65) imply that $u_{\xi,f}$ as defined in (2.3.52) is a function in $C_{\{0\}}$. Because A is continuous and bounded, and because $u_{\xi,f}$ satisfies the inhomogeneous equation (2.3.3), the derivative of $u_{\xi,f}$ is thus also in $C_{\{0\}}$. This shows that $u_{\xi,f} \in C^1_{\{0\}}$, by definition of $C^1_{\{0\}}$. This completes the argument that $f \in \operatorname{rge} D_A$. Thus D_A is surjective. □

The following corollary shows that in the presence of an exponential dichotomy for A, the Fredholm index of $\Lambda_{A,P}$ depends only on the rank of P.

Corollary 2.3.15. *Suppose that the bounded, continuous matrix valued function $A = A(t)$ admits an exponential dichotomy on $[0,\infty)$, with projection Π. Let P be any given projection on $\mathbb{R}^d$. Then the linear map $\Lambda_{A,P} : C^1_{\{0\}}\big([0,\infty),\mathbb{R}^d\big) \to C_{\{0\}}\big([0,\infty),\mathbb{R}^d\big)\times\operatorname{rge} P$ is Fredholm of index $k = \dim\operatorname{rge}\Pi - \dim\operatorname{rge} P$. In particular, if Π and P are of the same rank, then $\Lambda_{A,P}$ is Fredholm of index zero.*

Proof. According to Theorem2.3.14, the operator

$$D_A : C^1_{\{0\}} \to C_{\{0\}} \tag{2.3.66}$$

is Fredholm of index $\dim \operatorname{rge} \Pi$. Appending the zero space to the target space affects neither the kernel nor the co-dimension of the range, so the operator

$$(D_A,\, 0) : C^1_{\{0\}} \to C_{\{0\}} \times \{0\} \tag{2.3.67}$$

is also Fredholm of index $\dim \operatorname{rge} \Pi$. Enlarging the target space will increase the co-dimension of $\operatorname{rge}(D_A,\, 0)$ by a corresponding amount. Hence, the operator

$$(D_A,\, 0) : C^1_{\{0\}} \to C_{\{0\}} \times \operatorname{rge} P \tag{2.3.68}$$

is Fredholm of index $\dim \operatorname{rge} \Pi - \dim \operatorname{rge} P$. Notice that

$$\Lambda_{A,\,P} - (D_A,\, 0) = (0,\, P) \tag{2.3.69}$$

has finite rank, and is hence compact. Thus, the invariance of the Fredholm index under compact perturbations (Property 1.4.5 on page 10) implies that

$$\Lambda_{A,\,P} : C^1_{\{0\}} \to C_{\{0\}} \times \operatorname{rge} P \tag{2.3.70}$$

is Fredholm of index $\dim \operatorname{rge} \Pi - \dim \operatorname{rge} P$, as desired. □

The next corollary shows that the choice $P = \Pi$ yields an isomorphism, instead of just a map of index zero. This fact will be of limited use to us, because the nature of the problem under study is that the projection P is given, and because it is difficult (except in certain trivial cases) to identify the projection Π of the exponential dichotomy.

Corollary 2.3.16. *If the the bounded, continuous matrix valued function $A = A(t)$ admits an exponential dichotomy on $[0, \infty)$, with projection Π, then $\Lambda_{A,\Pi}$ is an isomorphism of $C^1_{\{0\}}\big([0, \infty), \mathbb{R}^d\big)$ onto $C_{\{0\}}\big([0, \infty), \mathbb{R}^d\big) \times \operatorname{rge}\Pi$.*

Proof. According to (2.3.39) in the theorem, any nonzero function in $\ker D_A$ takes a nonzero value in $\operatorname{rge}\Pi$ at $t = 0$, and hence is not in the kernel of $\Lambda_{A,\Pi}$. So $\ker \Lambda_{A,\Pi} = \{0\}$. To show that $\Lambda_{A,\Pi}$ is onto, let $f \in C_{\{0\}}$ and $\eta \in \operatorname{rge}\Pi$. By the theorem, D_A is surjective; choose $v_1 \in D_A^{-1}(\{f\})$. According once again to (2.3.39) in the theorem, there is some (unique) $v_2 \in \ker D_A$ such that $v_2(0) = \eta - \Pi v_1(0)$. It is easily seen that $\Lambda_{A,\Pi}(v_1 + v_2) = (f, \eta)$. □

In fact, the corollary can be strengthened optimally as follows. The difficulty in finding Π means that even this result is of limited practical use.

Corollary 2.3.17. *Suppose that the bounded, continuous matrix valued function $A = A(t)$ admits an exponential dichotomy on $[0, \infty)$, with projection Π. Let P be any given projection on $\mathbb{R}^d$ with the same rank as Π. Then $\Lambda_{A,P} : C^1_{\{0\}}\big([0, \infty), \mathbb{R}^d\big) \to C_{\{0\}}\big([0, \infty), \mathbb{R}^d\big) \times \operatorname{rge} P$ is an isomorphism if and only if $\operatorname{rge}\Pi \cap \ker P = \{0\}$.*

Proof. According to Corollary 2.3.15, $\Lambda_{A,P} : C^1_{\{0\}} \to C_{\{0\}} \times \operatorname{rge} P$ is Fredholm of index zero. Hence the following equivalences hold:

$$\Lambda_{A,P} \text{ is an isomorphism} \iff \ker \Lambda_{A,P} = \{0\} \tag{2.3.71}$$

$$\iff \ker D_A \cap \{u \colon u(0) \in \ker P\} = \{0\}. \tag{2.3.72}$$

As we have seen, $\ker D_A$ consists of all functions u of the form $u(t) = U(t)\xi$, where $U(t)$ is the fundamental matrix solution and $\xi \in \operatorname{rge}\Pi$. Hence,

$$\ker D_A \cap \{u \colon u(0) \in \ker P\} = \{0\} \iff \operatorname{rge}\Pi \cap \ker P = \{0\}, \tag{2.3.73}$$

which completes the proof. $\square$

Now we develop a converse to Theorem 2.3.14. This is done primarily for completeness; our main interest was in finding necessary conditions for $\Phi_{F,\,P}$ to be Fredholm of index zero. Still, it is of some interest that the exponential dichotomy criterion cannot be weakened. Also, we will make use of the converse to Theorem 2.3.14 to relate our earlier work with properness to the Fredholm properties of $\Lambda_{A,\,P}$. This will give a new necessary condition for the Fredholm property. See Section 2.3.5.

The converse to Theorem 2.3.14 was essentially proved by Palmer in [Pal88], except that Palmer views D_A as acting from $C_b^1\big([0,\,\infty)\big)$ into $C_b\big([0,\,\infty)\big)$. Also, we feel that in Palmer's proof, the application of either Theorem 64.B in [MS66] or the proof of Proposition 3 in [Cop78] (we are left to choose) is less than straightforward. For these reasons, we show how to reduce the problem to a straightforward application of the following, which is Proposition 3 on page 22 of [Cop78]:

Lemma 2.3.18. *Suppose that $A : [0,\,\infty) \to \mathbb{R}^{d\times d}$ is bounded and continuous. Then the inhomogeneous equation (2.3.3) has a bounded solution for every $f \in C_b\big([0,\,\infty),\,\mathbb{R}^d\big)$ if and only if the homogeneous equation (2.3.2) admits an exponential dichotomy.*

Our task is thus to show that the condition of this result of Coppel is satisfied whenever D_A is Fredholm as an operator from $C^1_{\{0\}}$ into $C_{\{0\}}$. In fact, we will do this under the seemingly weaker assumption that the range of D_A is closed in $C_{\{0\}}$. We begin, as did Palmer in [Pal88], with the following lemma:

Lemma 2.3.19. *Continue to assume that the matrix function A is continuous, and suppose that the range of $D_A : C^1_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big) \to C_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big)$ is closed in $C_{\{0\}}$. Then D_A is* *onto* $C_{\{0\}}$.

Proof. Since $C_0\big([0,\,\infty)\big)$ is a dense subspace of $C_{\{0\}}$, it suffices to show that the range of D_A contains $C_0\big([0,\,\infty)\big)$. Indeed, let $f \in C_0\big([0,\,\infty)\big)$, and suppose that $T > 0$ is such that f is supported in $[0,\,T]$. Then, for all $t > T$, we have

$$\int_0^t U(s)^{-1} f(s)\,\mathrm{d}s = \int_0^T U(s)^{-1} f(s)\,\mathrm{d}s. \tag{2.3.74}$$

In particular, if we take

$$\xi := -\int_0^T U(s)^{-1} f(s)\,\mathrm{d}s, \tag{2.3.75}$$

then the solution

$$u_{\xi,\,f}(t) = U(t)\big(\xi + \int_0^t U(s)^{-1} f(s)\,\mathrm{d}s\big) \tag{2.3.76}$$

is continuous and is supported in $[0,\,T]$. In particular, this solution is in $C^1_{\{0\}}$, so that f is in $\operatorname{rge} D_A$, as desired. □

Next, to prepare to construct a solution when $f \in C_b\big([0,\,\infty)\big)$, we prove a continuity property for the solutions u with respect to the function f. Now the solutions need not be unique, so we first need a consistent way to select one of them. (In this, and in the lemma which follows, we are following the development of Proposition 4 on page 22 of [Cop78].) Let V_1 denote the subspace of $\mathbb{R}^d$ consisting of all initial values for bounded solutions u for the homogeneous equation (2.3.2). Let V_2 be any subspace complementary to V_1. Let Π denote the linear projection onto V_1 along V_2. We then have

Lemma 2.3.20. *Continue to assume that A is a continuous matrix function, and suppose that D_A maps $C^1_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big)$ onto $C_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big)$. Then there exists $r = r(\Pi) > 0$ such* *that for every $f \in C_{\{0\}}$, the unique bounded solution u of (2.3.3) such that $u(0) \in V_2$ satisfies*

$$\left\|u\right\|_\infty \leq r \left\|f\right\|_\infty . \tag{2.3.77}$$

Proof. For each $f \in C_{\{0\}}$, the existence of a bounded solution $v \in C^1_{\{0\}}$ follows from the assumption of surjectivity. By definition of V_1, there is a unique bounded solution w to the homogeneous equation such that $w(0) = -\Pi v(0)$. Then, for $u = v + w$, u satisfies the inhomogeneous equation (2.3.3) for f, and $u(0) = v(0) - \Pi v(0) \in V_2$. To see that u is unique, the difference of two such solutions is a bounded solution to the homogeneous equation, whose initial value lies in V_2. Hence, by definition of V_2, that initial value is 0. Denote by S the map $f \mapsto u$.

It is readily checked that S is linear. It remains only to show that S is bounded as a map from $C_{\{0\}}$ into $C_b\big([0, \infty)\big)$; to do so we apply the closed graph theorem. Suppose that $f_n \to f$ in $C_{\{0\}}$ and $u_n = Sf_n \to u$ in $C_b\big([0, \infty)\big)$. Then

$$u(0) = \lim_{n\to\infty} u_n(0) \in V_2. \tag{2.3.78}$$

Hence, if u solves the inhomogeneous equation (2.3.3) for f, then $u = Sf$. Indeed, for any fixed t,

$$\lim_{n\to\infty} \int_0^t f_n(s)\,\mathrm{d}s = \int_0^t f(s)\,\mathrm{d}s \tag{2.3.79}$$

by uniform convergence. Hence,

$$\begin{aligned} u(t) - u(0) &= \lim_{n\to\infty} \int_0^t \dot{u}_n(s)\,\mathrm{d}s && (2.3.80)\\ &= \lim_{n\to\infty} \int_0^t -A(s)u_n(s) + f_n(s)\,\mathrm{d}s && (2.3.81)\\ &= \int_0^t -A(s)u(s) + f(s)\,\mathrm{d}s, && (2.3.82) \end{aligned}$$

again by uniform convergence. (Here we have used the uniform continuity of A on the compact $[0,\, t]$.) Thus, $u = Sf$, as desired. $\square$

We are now ready to state and prove a converse to Theorem 2.3.14:

Theorem 2.3.21. *Assume that the matrix function A is bounded and continuous. Assume that the range of the operator $D_A\colon C^1_{\{0\}}\big([0,\, \infty),\, \mathbb{R}^d\big) \to C_{\{0\}}\big([0,\, \infty),\, \mathbb{R}^d\big)$ is closed in $C_{\{0\}}$. Then the equation (2.3.2) admits an exponential dichotomy.*

Proof. To apply Lemma 2.3.18, let $f \in C^1_b$ be given. For each $n \in \mathbb{N}$, let f_n be a continuous function on $[0,\, \infty)$ which agrees with f on $[0,\, n]$, is supported in $[0,\, n{+}1]$, and which satisfies $\|f_n\|_\infty \le \|f\|_\infty$. By Lemma 2.3.19, D_A is onto $C_{\{0\}}$, whence Lemma 2.3.20 applies. Hence, let u_n denote the unique bounded solution of the inhomogeneous equation (2.3.3) for f such that $u_n(0) \in V_2$. Lemma 2.3.20 implies that for all $n \in \mathbb{N}$,

$$\|u_n\|_\infty \le r\,\|f_n\|_\infty \le r\,\|f\|_\infty\,. \tag{2.3.83}$$

One implication is that the sequence

$$\xi_n := u_n(0) \in V_2 \tag{2.3.84}$$

of initial values is bounded; hence there is a subsequence (ξ_{n_k}) and some $\xi \in V_2$ such that $v_{n_k} \to \xi$. We define $u : [0,\, \infty) \to \mathbb{R}^d$ by

$$u(t) := U(t)\big(\xi + \int_0^t U(s)^{-1} f(s)\,\mathrm{d}s\big). \tag{2.3.85}$$

Clearly, u so defined solves the inhomogeneous equation (2.3.2) for f; to apply Lemma 2.3.18, we must show that u is bounded. Fix $t \ge 0$. For $n > t$, $f_n(t) = f(t)$. Hence,

$$u_n(t) - u(t) = U(t)(\xi_n - \xi), \tag{2.3.86}$$

so that $u_{n_k}(t) \to u(t)$ as $k \to \infty$. In particular, $\left|u(t)\right| =\leq r \left\|f\right\|_\infty$. Since $t \geq 0$ was arbitrary, u is indeed bounded. We conclude from Lemma 2.3.18 that the homogeneous equation (2.3.2) admits an exponential dichotomy, as desired. $\square$

To summarize, we present the following theorem.

Theorem 2.3.22. *Assume that A is a continuous, bounded matrix function from $[0, \infty)$ to $\mathbb{R}^{d\times d}$. Define the continuous linear operator*

$$D_A : C^1_{\{0\}}\big([0, \infty), \mathbb{R}^d\big) \to C_{\{0\}}\big([0, \infty), \mathbb{R}^d\big) \tag{2.3.87}$$

by the equation

$$D_A u(t) := \dot{u}(t) + A(t)u(t). \tag{2.3.88}$$

Then the following are equivalent:

1. *The homogeneous equation (2.3.2) admits an exponential dichotomy (with some associated projection Π) over $[0, \infty)$ [and $\dim \operatorname{rge} \Pi = n$].*
2. *The operator D_A is surjective [and $\dim \ker D_A = n$].*
3. *The operator D_A is Fredholm [of index n].*
4. *The range of D_A is closed in $C_{\{0\}}$.*
5. *For every $f \in C_b\big([0, \infty)\big)$, there is a bounded solution to the inhomogeneous equation (2.3.3) for f.*

Proof. Theorem 2.3.14 proves that the first statement implies the second. The second implies the third by definition, the third implies the fourth by definition, and the fourth implies the first by Theorem 2.3.21. The equivalence of the fifth statement with the first statement is Lemma [Cop78]. $\square$

Corollary 2.3.23. *Assume that A is a continuous, bounded matrix function from $[0,\, \infty)$ to $\mathbb{R}^{d\times d}$, and let P be a given projection on $\mathbb{R}^d$. Define the continuous linear operator*

$$\Lambda_{A,P} : C^1_{\{0\}}\big([0,\, \infty),\, \mathbb{R}^d\big) \to C_{\{0\}}\big([0,\, \infty),\, \mathbb{R}^d\big) \times \operatorname{rge} P \tag{2.3.89}$$

by the equation

$$\Lambda_{A,\,P} u := \big(D_A u,\; Pu(0)\big). \tag{2.3.90}$$

Then the following are equivalent:

1. *The homogeneous equation (2.3.2) admits an exponential dichotomy over $[0,\, \infty)$ with some associated projection Π of rank k.*
2. *The operator $\Lambda_{A,\,P}$ is Fredholm of index $k - \dim \operatorname{rge} P$.*

Proof. That the first statement implies the second is just Corollary 2.3.15. The reverse implication is proved as follows. If $\Lambda_{A,\,P}$ is Fredholm of index $k - \dim \operatorname{rge} P$ from $C^1_{\{0\}}$ into $C_{\{0\}} \times \operatorname{rge} P$, then D_A is Fredholm of index k from $C^1_{\{0\}}$ into $C_{\{0\}}$. This can be seen by reversing the calculations of the proof of Corollary 2.3.15. Thus, Theorem 2.3.22 implies that the first statement is true, as desired. □

Example 2.3.24. Here we examine in detail the Fredholm property in case $A(t) = A_0$ is a constant. We refer to Example 2.3.7 for discussion of exponential dichotomy in this case. First, suppose that A_0 has no eigenvalues on the imaginary axis, and let k be the algebraic count of those eigenvalues of A_0 that have positive real part. According to Example 2.3.7, A_0 admits an exponential dichotomy with projection Π of rank k. Thus, according to Corollary 2.3.15, the map $\Lambda_{A_0,\,P}$ is Fredholm of index zero from $C^1_{\{0\}}$ into $C_{\{0\}} \times \operatorname{rge} P$ if and only if the rank of P is k.

Now suppose that $\Lambda_{A_0,\,P}$ is indeed Fredholm of index zero from $C^1_{\{0\}}$ into $C_{\{0\}} \times \operatorname{rge} P$, and moreover that the $\ker P$ contains no nontrivial generalized eigenvectors of A_0 corresponding to eigenvalues with positive real part. Then, according to Corollary 2.3.17, $\Lambda_{A_0,\,P}$ is then an isomorphism.

On the other hand, if A_0 has any pure imaginary eigenvalues, then A_0 does not admit an exponential dichotomy. Thus, Corollary 2.3.23 implies that $\Lambda_{A,\,P}$ is not Fredholm of any index nor an isomorphism, for any projection P.[10] ♦

Example 2.3.25. Suppose that $A = A(t)$ is p-periodic. This may be viewed as a continuation of Example 2.3.8, where the exponential dichotomy property was studied. Suppose that $U(p)$ has no eigenvalues of modulus 1, and let k be the algebraic count of those eigenvalues of $U(p)$ that have modulus greater than 1. According to Example 2.3.8, A admits an exponential dichotomy with projection Π of rank k. Thus, according to Corollary 2.3.15, the map $\Lambda_{A,\,P}$ is Fredholm of index zero from $C^1_{\{0\}}$ into $C_{\{0\}} \times \operatorname{rge} P$ if and only if the rank of P is k.

Now suppose that $\Lambda_{A,\,P}$ is indeed Fredholm of index zero from $C^1_{\{0\}}$ into $C_{\{0\}} \times \operatorname{rge} P$, and moreover that the $\ker P$ contains no nontrivial generalized eigenvectors of $U(p)$ corresponding to eigenvalues with real part greater than 1. Then, according to Corollary 2.3.17, $\Lambda_{A,\,P}$ is then an isomorphism.

On the other hand, if $U(p)$ has any eigenvalues of modulus 1, then A does not admit an exponential dichotomy. Thus, Corollary 2.3.23 implies that $\Lambda_{A,\,P}$ is not Fredholm of any index nor an isomorphism, for any projection P.[11] ♦

Because of Corollary 2.3.23, any known sufficient condition for exponential dichotomy

[10] In fact, Theorem 2.3.22 implies that the range of D_A is not even closed in $C_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big)$.

[11] Once again, Theorem 2.3.22 implies that the range of D_A is not even closed in $C_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big)$.

serves as a sufficient condition for the Fredholm property. For example, in Section 6 of [Cop78], Coppel gives a diagonal dominance criterion for exponential dichotomy. Using this with Corollary 2.3.23 immediately gives the following condition:

Theorem 2.3.26. *Assume that $A = A(t) = (a_{ij}(t))$ is a continuous, bounded matrix function from $[0, \infty)$ to $\mathbb{R}^{d\times d}$, and let P be a given projection on $\mathbb{R}^d$. Suppose there exists $\delta > 0$ such that*

$$\bigl|a_{ii}(t)\bigr| \geq \delta + \sum_{\substack{j=1 \\ j\neq i}} \bigl|a_{ij}(t)\bigr| \tag{2.3.91}$$

for all $t \geq 0$ and all $i = 1, \ldots, d$.

Then the operator $\Lambda_{A,P} : C^1_{\{0\}}\bigl([0, \infty), \mathbb{R}^d\bigr) \to C_{\{0\}}\bigl([0, \infty), \mathbb{R}^d\bigr) \times \operatorname{rge} P$ is Fredholm of index $k - \dim \operatorname{rge} P$, where k is the number of indices i such that $a_{ii}(t) > 0$.

The next result is taken from Coppel[12] [Cop78], Section 7. Let $A = A(t)$ be a bounded and continuous matrix function on $[0, \infty)$, and let $H = H(t)$ be a bounded, continuously differentiable, Hermetian matrix function. Let $\psi = \psi(t, x)$ denote the quadratic form associated with $H(t)$:

$$\psi(t, x) := \bigl\langle H(t)x, x\bigr\rangle . \tag{2.3.92}$$

Then ψ is said to be a *Lyapunov function* for the differential equation

$$\dot{u}(t) + A(t)u(t) = 0 \tag{2.3.93}$$

if its time derivative is negative-definite along solutions of (2.3.93), *i.e.* if there exists $\beta > 0$ such that

$$\overset{\bullet}{\overbrace{\psi\bigl(t, u(t)\bigr)}} \leq -\beta \bigl|u(t)\bigr|^2 . \tag{2.3.94}$$

[12] Coppel uses $-A$ instead of A.

By use of (2.3.93), one finds that (2.3.94) is equivalent to

$$\Big\langle \big(\dot{H}(t) - H(t)A(t) - A(t)^* H(t)\big)u(t),\, u(t) \Big\rangle \leq -\beta \left|u(t)\right|^2. \tag{2.3.95}$$

Equivalently,

$$H(t)A(t) + A(t)^* H(t) - \dot{H}(t) \geq \beta I \qquad \text{(as quadratic forms).} \tag{2.3.96}$$

That such H exists turns out to be equivalent to the admission of an exponential dichotomy. We have added the third item, as justified by Corollary 2.3.23.

Lemma 2.3.27 (Coppel[Cop78]). *Let $A\colon [0,\,\infty] \to \mathbb{R}^{d\times d}$ be bounded and continuous. The following are equivalent:*

1. *A admits an exponential dichotomy on $[0,\,\infty)$ with an associated projection of some rank k.*
2. *There exists a bounded, continuously differentiable, Hermetian matrix function $H = H(t)$ that satisfies (2.3.96).*
3. *For each projection P on $\mathbb{R}^d$ of rank k, the operator*

$$\Lambda_{A,\,P}\colon C^1_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big) \to C_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big) \times \operatorname{rge} P$$

   is Fredholm of index 0.

2.3.5 Properness and the Fredholm property

Consider the case that the limit $A_0 = \lim_{t\to\infty}$ exists. By showing that the operator $D_A - D_{A_0}$ is compact from $C^1_{\{0\}}$ into $C_{\{0\}}$, it is possible to show that the Fredholm property and index of $\Lambda_{A,\,P}$ agree with those of $\Lambda_{A_0,\,P}$. A similar statement can be made for asymptotically periodic matrices. In both situations, notice that we are effectively replacing A by the member(s) of $\omega(A)$ to determine the Fredholm property and index. This is not surprising, because the exponentially dichotomous properties of A depend only on the values of $A(t)$ on any interval $[T,\,\infty)$. However, the compactness argument will fail in general. This is because different members B and B' of $\omega(A)$ need not induce operators D_B and $D_{B'}$ which are compact perturbations of one another; in such a case it is clearly unreasonable to expect D_B to be a compact perturbation of D_A.

However, our earlier results on properness provide clues about how to relate $\omega(A)$ to A. Recall Yood's criterion (Property 1.4.9) for properness on closed bounded sets. Yood's criterion provides a useful connection between Theorems 2.2.26 and 2.3.22, as follows.

Lemma 2.3.28. *Suppose that $A = A(t)$ is a bounded, uniformly continuous $\mathbb{R}^{d\times d}$-valued function on $[0,\,\infty)$. Also assume that for all $B \in \omega(A)$, the homogeneous linear equation*

$$\dot{u}(t) + B(t)u(t) = 0 \tag{2.3.97}$$

has no nontrivial solution bounded on $\mathbb{R}$.

Then the operator $\Lambda_{A,\,P}$ is Fredholm of index $\dim\ker D_A - \dim \operatorname{rge} P$ as a map from $C^1_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big)$ into $C_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big) \times \operatorname{rge} P$.

Proof. According to Theorem 2.2.26, the operator $\Lambda_{A,P}$ is proper on the closed bounded subsets of $C^1_{\{0\}}$. Thus, Yood's criterion implies that the range of $\Lambda_{A,P}$ is closed in $C_{\{0\}} \times \operatorname{rge} P$. In particular, the range of D_A is closed in $C_{\{0\}}$. Thus, according to Theorem 2.3.22, the matrix A admits an exponential dichotomy. We conclude from Corollary 2.3.23 that $\Lambda_{A,P}$ is Fredholm of index $n - \dim \operatorname{rge} P$, where n is the rank of a projection associated with this exponential dichotomy.

By Theorem 2.3.22, $n = \dim \ker D_A$. This completes the proof. □

Notice that one cannot conclude from the preceding lemma that the Fredholm index of $\Lambda_{A,P}$ is zero, unless one knows the dimension of D_A. It would be helpful to be able to deduce this information from the structure of $\omega(A)$. This information turns out to be available, because there is already much in the literature about $\omega(A)$ when A is linear. We refer in particular to the paper [Sac79] of Sacker. Several comments must be made concerning this paper, in which Sacker explores the relationship between (i) the Fredholm property and index of an operator like D_A, and (ii) the solutions to equations in its "α- and ω-limit sets". The main result of the paper, called Theorem 3, is similar in spirit to Theorem 2.3.30, below. However, there are two important differences. First, the Banach spaces under consideration by Sacker are spaces of functions that share some *local* property. For example, instead of using $C_{\{0\}}$ as a target space, Sacker uses the space of bounded continuous functions, topologized by uniform convergence on compact sets. This space is larger than $C_{\{0\}}$ and is equipped with a weaker topology.[13] The admissible domain space is consequently enlarged; it is the space of bounded, uniformly continuous functions. This space does not incorporate our desired boundary condition at infinity. Sacker considers other "local" spaces as well,

[13] Consider a sequence of "bump" functions, where the bump slides off to infinity.

such as L^p_{loc}; all of them involve a local definition.

A second, less fundamental difference is that Sacker uses spaces of functions that are defined on all of $\mathbb{R}$, while we are working on $[0, \infty)$.

Given these remarks, we cannot apply directly any of the results from [Sac79] due to the difference of setting. Instead, we start from Sacker's four remarks that start near equation (1.10) of [Sac79]. In these remarks, Sacker summarizes some results from three earlier papers ([SS74] and [SS76] with Sell, and also [Sac78]). We need only the following result, contained in Remark 4 of [Sac79].

Lemma 2.3.29. *Suppose that $A = A(t)$ is a bounded, uniformly continuous $\mathbb{R}^{d\times d}$-valued function on $[0, \infty)$. Also assume that for no $B \in \omega(A)$ does the homogeneous linear equation*

$$\dot{u}(t) + B(t)u(t) = 0 \tag{2.3.98}$$

have a nontrivial solution bounded on $\mathbb{R}$.

Then, for all $B \in \omega(A)$,

$$\dim \ker D_B = \dim \ker D_A, \tag{2.3.99}$$

where D_B is here understood to act on $C^1_{\{0\}}([0, \infty))$ (not $C^1_{\{0\}}(\mathbb{R})$).

We can thus refine Lemma 2.3.28 as follows:

Theorem 2.3.30. *Suppose that $A = A(t)$ is a bounded, uniformly continuous $\mathbb{R}^{d\times d}$-valued function on $[0, \infty)$. Also assume that for no $B \in \omega(A)$ does the homogeneous linear equation*

$$\dot{u}(t) + B(t)u(t) = 0 \tag{2.3.100}$$

have a nontrivial solution bounded on $\mathbb{R}$.

Let B be any member of $\omega(A)$, and let $k = \dim\ker D_B$, where D_B is understood to act on $C^1_{\{0\}}([0,\infty))$ (not $C^1_{\{0\}}(\mathbb{R})$). Then the operator $\Lambda_{A,P}$ is Fredholm of index $k - \dim\operatorname{rge} P$ as a map from $C^1_{\{0\}}$ into $C_{\{0\}} \times \operatorname{rge} P$.

Proof. The hypotheses are exactly as in Lemma 2.3.28, so we know that the operator $\Lambda_{A,P}$ is Fredholm of index $\dim\ker D_A - \dim\operatorname{rge} P$. We then use Lemma 2.3.29 to replace $\dim\ker D_A$ by $\dim\ker D_B$. □

Now we show how to use this result to handle the cases mentioned earlier. As mentioned, these are also easily handled by a compactness argument, but our technique will be seen to work in situations where compactness fails.

Example 2.3.31. Suppose that $A_0 = \lim_{t\to\infty} A(t)$ exists, and that P is a given projection on $\mathbb{R}^d$. Then, according to Example 2.2.24 (or by a simple verification), $\omega(A) = \{A_0\}$. That there be no nontrivial solution to $\dot{u} + A_0 u = 0$ that is bounded on $\mathbb{R}$ is equivalent to the nonexistence of imaginary eigenvalues of A_0. Also, the dimension k of $\ker A_0$ in $C^1_{\{0\}}\big([0,\infty)\big)$ is the same as the algebraic number of eigenvalues of A_0 with positive real part. Hence, $\Lambda_{A,P}$ is Fredholm of index $k - \dim\operatorname{rge} P$ as a map from $C^1_{\{0\}}$ into $C_{\{0\}} \times \operatorname{rge} P$, as long as A_0 has no imaginary eigenvalues. ◆

Example 2.3.32. Suppose A is asymptotically periodic, so that there is a p-periodic function $B = B(t)$ such that $\lim_{t\to\infty} \big|A(t) - B(t)\big|$. Then, according to Example 2.2.24,

$$\omega(A) = \{\tau_\sigma B\colon 0 \le \sigma < p\}\,. \tag{2.3.101}$$

It is clear that the hypotheses of Theorem 2.3.30 are met so long as there are no nontrivial solutions bounded on $\mathbb{R}$ to the single equation $\dot{u} + B(t)u = 0$. By the Floquet theory, this is the case as long as $U(p)$ has no eigenvalues of modulus 1, where U is the fundamental matrix solution for B. In this case, the dimension of $\ker D_B$ in $C^1_{\{0\}}\big([0,\infty)\big)$ is the same as the algebraic number of eigenvalues of $U(p)$ of modulus less than 1, as discussed in earlier examples. ♦

Next, for a more sophisticated example, we turn to the asymptotically constant functions, as introduced in Definition 2.2.19 of Section 2.2.4.

Example 2.3.33. Assume that A is asymptotically constant, meaning that $\omega(A)$ consists entirely of constant functions. The condition that (2.3.100) have no nontrivial solution bounded on $\mathbb{R}$ thus reduces to the condition that for no $B \in \omega(A)$ does B have eigenvalues on the imaginary axis. In this case, the conclusion is that $\Lambda_{A,P} : C^1_{\{0\}}\big([0,\infty)\big) \to C_{\{0\}}\big([0,\infty)\big) \times \operatorname{rge} P$ is Fredholm of index $k - \dim \operatorname{rge} P$, where k is algebraic number of eigenvalues of B with positive real part, for any chosen $B \in \omega(A)$. Put another way, the eigenvalues of $A = A(t)$ are eventually bounded away from the imaginary axis, and k is the algebraic count of those eigenvalues which stay to the right of the imaginary axis. ♦

Remark 2.3.34. Example 2.3.33 could also be obtained by use of Proposition 1 in Coppel [Cop78], which says essentially the same thing in terms of exponential dichotomies instead of the Fredholm property. However, Theorem 2.3.30 is not limited to this kind of application. See Section 2.5.3 for a different example of the use of Theorem 2.3.30.

2.3.6 Relevance to nonzero degree

As mentioned at the start of Section 2.3.3, the knowledge that $D\Phi_{F,P}(0)$ is an isomorphism is relevant to another part of the degree argument. The following remarks depend on the properties of the absolute degree that are discussed starting on page 12. We will want to know that for some ball B_R centered at $0 \in X = C^1_{\{0\}}$, the absolute degree $\left|d\right| \left(\Phi_{F,\,P},\, B_R,\, (0,0)\right)$ is well-defined and is nonzero. Hence, the knowledge that $D\Phi_{F,\,P}(0)$ is an isomorphism is of use in satisfying the hypotheses of Property 1.5.6 of the absolute degree.

If this were the only practical way to tell that $\left|d\right| \left(\Phi_{F,\,P},\, B_R,\, (0,0)\right) \neq 0$, then we would have to insist that $D\Phi_{F,\,P}(0)$ be an isomorphism, which would devalue all of the other ways that we have to ensure that $\Phi_{F,\,P}$ is Fredholm of index zero. This is not the case, thanks to Borsuk's Theorem (Property 1.5.7 on page 14). Also, the hypotheses of Borsuk's Theorem are very easy to satisfy. We record the following, for later use:

Lemma 2.3.35. *Let Ξ be defined as in Section 1.5 with respect to the Banach spaces $X = C^1_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big)$ and $Y = C_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big)$. Let B_R be the open ball of radius R centered at $0 \in C^1_{\{0\}}$. Let P be a projection on $\mathbb{R}^d$, and suppose that $F\colon [0,\,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is such that $\big(\Phi_{F,\,P},\, B_R,\, (0,0)\big) \in \Xi$, so that $\left|d\right| \big(\Phi_{F,\,P},\, B_R,\, (0,0)\big)$ is well-defined.*

If F is such that $F(t,\,-x) = -F(t,\,x)$, then $\left|d\right| \big(\Phi_{F,\,P},\, B_R,\, (0,0)\big) \neq 0$.

Proof. According to Borsuk's Theorem, it suffices to verify that $\Phi_{F,\,P}(-u) = -\Phi_{F,\,P}(u)$. Indeed,

$$\Phi_{F,\,P}(-u)(t) = \Big(-\dot{u}(t) + F\big(t,\,(-u)(t)\big),\; P(-u)(0)\Big) \tag{2.3.102}$$

$$= \Big(-\dot{u}(t) - F\big(t,\,u(t)\big),\; -Pu(0)\Big) \tag{2.3.103}$$

$$= -\Phi F,\, P(u)(t), \tag{2.3.104}$$

□

as desired.

2.4 AN EXISTENCE THEOREM

Let us briefly review the work so far. Let X be the Banach space $C^1_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big)$. Let a function $F\colon [0,\,\infty)\times\mathbb{R}^d \to \mathbb{R}^d$ and a projection P on $\mathbb{R}^d$ be given. Let Y be the Banach space $C_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big)\times \operatorname{rge} P$. Under appropriate conditions on F, we have shown that for each $f\in C_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big)$ and $\xi\in\operatorname{rge} P$, the existence of a solution $u\in C^1\big([0,\,\infty),\,\mathbb{R}^d\big)$ to the boundary value problem

$$\begin{cases} \dot{u}(t) + F\big(t,\,u(t)\big) = f(t), \qquad \forall t\ge 0,\\ Pu(0) = \xi,\\ \lim\limits_{t\to\infty} u(t) = 0 \end{cases} \tag{2.4.1}$$

is equivalent to the existence of a solution $u\in C^1_{\{0\}}$ to the functional equation

$$\Phi_{F,\,P}(u) = (f,\,\xi). \tag{2.4.2}$$

Section 1.5 outlines one way to do this, but the technique demands that we know that the operator $\Phi_{F,\,P}(u)$ is C^1, Fredholm of index zero, proper on the closure of a chosen bounded open subset of $C^1_{\{0\}}$, *et cetera*. Now that we have studied each of these properties with respect to our choice of $\Phi_{F,\,P}$, we are prepared to state and prove an existence result. Then, in Section 2.5, we will show that the existence theorem has hypotheses that are practical to verify in nontrivial examples.

Theorem 2.4.1. *Let a function $F\colon [0,\,\infty)\times\mathbb{R}^d\to\mathbb{R}^d$ and a projection P on $\mathbb{R}^d$ be given. Suppose that all of the following conditions hold:*

1. *Conditions (2.1.1a) – (2.1.1c) hold of F.* *(See page 16.)*
2. *The map F has an admissible omega set $\omega(F)$.* *(See Definition 2.2.13 on Page 37.)*

3. *The map $\Phi_{F,\,P}$ is Fredholm of index zero as an operator from $X = C^1_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big)$ into $Y = C_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big) \times \operatorname{rge} P$.*

4. *For a given nonzero pair $(f,\,\xi) \in C_{\{0\}} \times \operatorname{rge} P$, the solutions $u \in C^1_{\{0\}}$ to the initial value problem*

$$\left.\begin{aligned} \dot u(t) + F\big(t,\,u(t)\big) = sf(t) \qquad \forall t \geq 0 \\ Pu(0) = s\xi \end{aligned}\right\} \tag{2.4.3}$$

are bounded a priori *in $C^1_{\{0\}}$-norm by some $R > 0$, independent of $s \in [0,\,1]$.*

Let B be the open ball of radius $R+1$ centered at $0 \in C^1_{\{0\}}$. Then the absolute degree $\big|d\big|\,\big(\Phi_{F,\,P},\,B,\,(sf, s\xi)\big)$ is well-defined for all $s \in [0,\,1]$. If also:

5. *The absolute degree $\big|d\big|\,\big(\Phi_{F,\,P},\,B,\,(0,0)\big)$ is nonzero.*

Then the initial value problem (2.4.3) has (at least) a solution $u \in C^1_{\{0\}}$, for each $s \in [0,\,1]$.

Proof. According to Item 1 and Corollary 2.1.3, the map $\Phi_{F,\,P}$ is a well-defined C^1 map of $C^1_{\{0\}}$ into $C_{\{0\}} \times \operatorname{rge} P$. By assumption (as Item 3) the map is also Fredholm of index zero from $C^1_{\{0\}}$ into $C_{\{0\}} \times \operatorname{rge} P$. According to Item 2 and Theorem 2.2.14, Φ is proper on the closed ball $\overline{B}$. Hence, if $(g,\,\eta)$ is any point in $Y \setminus \Phi_{F,\,P}(\partial B)$, then

$$\big(\Phi_{F,\,P},\,B,\,(g,\eta)\big) \in \Xi, \tag{2.4.4}$$

as defined in Definition 1.5.1 on page 13.

Note that Item 4 implies that for all $s \in [0,\,1]$,

$$\Phi^{-1}_{F,\,P}\big(\{(sf,\,s\xi)\}\big) \subset B. \tag{2.4.5}$$

In particular,

$$(sf,\,s\xi) \in Y \setminus \Phi_{F,\,P}(\partial B) \quad \forall s \in [0,\,1]. \tag{2.4.6}$$

This proves the first conclusion, that the absolute degree $\left|d\right|\big(\Phi_{F,P},\,B,\,(sf,s\xi)\big)$ is well-defined for all $s\in[0,\,1]$.

We introduce the following homotopy $h\colon[0,\,1]\times C^1_{\{0\}}\to C_{\{0\}}\times\operatorname{rge}P$:

$$h(s,\,u):=\Phi_{F,P}(u)-(sf,\,s\xi). \tag{2.4.7}$$

Notice that $h(0,\,\cdot)=\Phi_{F,P}$, that $h(1,\,\cdot)=\Phi_{F,P}-(f,\,\xi)$, and that

$$h(s,\,u)=0\qquad\Longleftrightarrow\qquad\Phi_{F,P}(u)=(sf,\,s\xi). \tag{2.4.8}$$

That h is C^1 follows trivially from the fact (Corollary 2.1.3) that $\Phi_{F,P}$ is C^1. The properness of $h\big|_{[0,\,1]\times\overline{B}}$ results from the properness of $\Phi_{F,P}$ on $\overline{B}$ as follows. Assume that $(s_n,\,u_n)$ is a sequence in $[0,\,1]\times\overline{B}$ such that $\big(h(s_n,\,u_n)\big)$ is convergent in $C_{\{0\}}\times\operatorname{rge}P$, to some $(g,\,\eta)$. In any case, (s_n) has a convergent subsequence $s_{n_k}\to s_0\in[0,\,1]$. Thus,

$$\begin{aligned}\Phi_{F,P}(u_{n_k})&=h(s_{n_k},\,u_{n_k})+(s_{n_k}f,\,s_{n_k}\xi) &(2.4.9)\\ &\to(g,\,\eta)+(s_0f,\,s_0\xi)\text{ as }k\to\infty. &(2.4.10)\end{aligned}$$

The already established properness of $\Phi_{F,P}$ then implies that there is a convergent subsequence of $\big(u_{n_k}\big)$. This shows that $h\big|_{[0,\,1]\times\overline{B}}$ is proper. To see that h is Fredholm of index 1 from $[0,\,1]\times C^1_{\{0\}}$ into $C_{\{0\}}\times\operatorname{rge}P$, notice that for all $(s,\,u)$ and $(a,\,v)\in[0,\,1]\times C^1_{\{0\}}$,

$$Dh(s,\,u)(a,\,v)=D\Phi_{F,P}(u)v-a(f,\,\xi), \tag{2.4.11}$$

or in block matrix form,

$$Dh(s,\,u)=\Big((f\quad\xi)^T\;\Big|\;D\Phi_{F,P}(u)\Big) \tag{2.4.12}$$

Notice that the operator

$$L = \Big((0 \quad 0)^T \;\Big|\; D\Phi_{F,\,P}(u) \Big) \tag{2.4.13}$$

is Fredholm of index 1 because L has the same target space and range as $D\Phi_{F,\,P}(u)$, which has Fredholm index zero, but $\ker L = \mathbb{R} \times \ker D\Phi_{F,\,P}(u)$ is larger by one dimension. Since $Dh(s,\,u) - L$ is compact (even of finite rank), it follows from Property 1.4.5 on page 10 that $Dh(s,\,u)$ is Fredholm of index 1. Thus, h is Fredholm of index 1 by definition. All of this implies that we may use the homotopy invariance of the absolute degree (Property 1.5.4 on page 13) to conclude that

$$\big|d\big|\,\big(h(0,\,\cdot),\,B,\,(0,0)\big) = \big|d\big|\,\big(h(s,\,\cdot),\,B,\,(0,0)\big) = \big|d\big|\,\big(h(1,\,\cdot),\,B,\,(0,0)\big) \tag{2.4.14}$$

for all $s \in [0,\,1]$. As we have already noted, $h(0,\,\cdot) = \Phi_{F,\,P}$. With Item 5, we thus have

$$\big|d\big|\,\big(\Phi_{F,\,P} - (sf,\,s\xi),\,B,\,(0,0)\big) \neq 0. \tag{2.4.15}$$

Because of the normalization property of the absolute degree (Property 1.5.2 on page 13), this implies that there is some $u \in X = C^1_{\{0\}}$ such that

$$\Phi_{F,\,P}(u) = (sf,\,s\xi). \tag{2.4.16}$$

By definition of $\Phi_{F,\,P}$ (and see Remark 2.1.5) we conclude that the initial value problem (2.4.3) has a solution $u \in C^1_{\{0\}}$, for each $s \in [0,\,1]$, as desired. $\square$

Remark 2.4.2. Suppose that hypotheses 1, 2, and 4 in Theorem 2.4.1 are met. Let $A(t) := D_x F(t,\, 0)$. Suppose that the linear operator $\Lambda_{A,\,P} = D\Phi_{F,\,P}$ is an isomorphism of the Banach space $C^1_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big)$ onto $C_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big) \times \operatorname{rge} P$. Then hypothesis 3 of Theorem 2.4.1 is met as well. Suppose in addition that there are no nontrivial solutions in $C^1_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big)$ to the nonlinear homogeneous system $\Phi_{F,\,P}(u) = (0,\,0)$. Then we apply Property 1.5.6 of the absolute degree (on page 14) to conclude that the final hypothesis of Theorem 2.4.1 is met, as well. $\Diamond$

Remark 2.4.3. Suppose that hypotheses 1, 2, and 4 in Theorem 2.4.1 are met. Suppose that hypothesis 3 has also been verified, perhaps by use of the methods of Sections 2.3.4 or 2.3.5. Suppose in addition that F is odd in the variable x, *i.e.* that $F(t,\,-x) = -F(t,\,x)$. Then we apply Borsuk's Theorem 1.5.7 (on page 14; see also Lemma 2.3.35 on page 82) to conclude that the final hypothesis of Theorem 2.4.1 is met, as well. $\Diamond$

2.5 EXAMPLES

Remark 2.5.1. See Rabier and Stuart [RSb, RSa] for examples in case there is some function $F^\infty \in C^1(\mathbb{R}^d,\,\mathbb{R}^d)$ such that $F(0) = 0$ and $D_x F(t,\,x) \to DF^\infty(x)$ as $t \to \infty$, uniformly on bounded subsets of $\mathbb{R}^d$. Even though the setting is different there ($W^{1,p}$ instead of $C^1_{\{0\}}$ and L^p instead of $C_{\{0\}}$), there is enough similarity that we focus on examples where this is not assumed. In examples where F does have this special asymptotic behavior, the below arguments will be simplified.

2.5.1 A one-dimensional example

Let $F\colon [0,\,\infty)\times\mathbb{R}\to\mathbb{R}$ be given by

$$F(t,\,x) := g(t)p(x) := \big((\sin\sqrt{t})+2\big)(x+x^3). \tag{2.5.1}$$

We seek solutions in $C^1_{\{0\}}([0,\,\infty),\,\mathbb{R})$ to the initial value problem

$$\left.\begin{aligned} \dot{u}(t) + \big((\sin\sqrt{t})+2\big)\big(u(t)+u(t)^3\big) = f(t), \quad t\geq 0 \\ u(0)=\xi, \end{aligned}\right\} \tag{2.5.2}$$

where $f\in C_{\{0\}}([0,\,\infty),\,\mathbb{R})$ and $\xi\in\mathbb{R}$ are given.

Remark 2.5.2. This problem is covered by a more general example that will be discussed in Section 2.5.2 on page 93. We include this example separately because we feel it is instructive to verify the various hypotheses with particular functions. $\Diamond$

Remark 2.5.3. The choice of F may seem strange. In particular, one might wonder about the function $g(t)$. We do not want $g(t)$ to have a limit as $t\to\infty$, because this case is essentially covered by Rabier and Stuart [RSb, RSa], although in the Sobolev space setting. We do not want g to be periodic, for exactly the same reason. We also point out that for this choice of g, if we omit the "+2" from the expression, then we will have $0\in\omega(F)$, which would destroy the properness property. $\Diamond$

We will use Theorem 2.4.1 to prove that a solution to (2.5.2) exists in $C^1_{\{0\}}$. Because this is an initial value problem, the projection P is simply the identity function. To check the first item in Theorem 2.4.1, we check that F satisfies conditions (2.1.1a)–(2.1.1b) on page 16. For this, we note

$$D_xF(t,x) = g(t)Dp(x) = \big(\sin\sqrt{t}+2\big)(1+3x^2). \tag{2.5.3}$$

The continuity of F and D_xF are clear, which verifies condition (2.1.1a). In addition $F(t, 0) = g(t)0 = 0$, which verifies condition (2.1.1c). Notice that g is differentiable for $t > 0$, and

$$\dot{g}(t) = \frac{\cos(\sqrt{t})}{2\sqrt{t}}. \tag{2.5.4}$$

Thus, $\dot{g}$ is bounded on $[1, \infty)$ (in fact, $\dot{g}(t) \to 0$ as $t \to \infty$), so that g is uniformly continuous on $[1, \infty)$. Of course, g is already uniformly continuous on the compact interval $[0, 1]$. In view of equations (2.5.1) and (2.5.3), this verifies the third condition (2.1.1a), as well as the first item in Theorem 2.4.1.

Next, we consider the omega-limit set of F. Since $\dot{g}(t) \to 0$ as $t \to \infty$, we are in the situation of Remark 2.2.25 on page 49 (with $A = g$). Accordingly, $\omega(g)$ consists of an interval of constant functions equal to the range of the function $\sin + 2$:

$$\omega(g) = [1, 3] \subset C_b(\mathbb{R}). \tag{2.5.5}$$

Thus,

$$\omega(F) = \left\{G\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R} \colon G(t, x) = ax + ax^3 \text{ for some } 1 \le a \le 3\right\}. \tag{2.5.6}$$

From this, it is clear that $\omega_0(F) = \{0\}$ (see page (2.2.4) for the definition). Thus, $\omega_0(F)$ is totally disconnected, which is a requirement for the admissibility of $\omega_0(F)$. Suppose now that u is a non-constant C^1 solution of the equation

$$\dot{u}(t) + au(t) + au(t)^3 = 0, \tag{2.5.7}$$

where $1 \le a \le 3$. To show that u must be unbounded on $\mathbb{R}$, notice that if $u(t) > 0$, then equation (2.5.7) implies that $\dot{u}(t) < -u(t)$. This implies readily that $u(t) \to \infty$ as $t \to -\infty$. Similarly, if $u(t) < 0$ then $\dot{u}(t) > -u(t)$, whence $u(t) \to -\infty$ as $t \to -\infty$. This shows that $\omega(F)$ is admissible (see Definition 2.2.13 on page 37), and we have verified Item 2 in Theorem 2.4.1.

Notice that

$$D_x F(t,\, 0) = g(t), \tag{2.5.8}$$

and recall that $P = I$ in this example. Hence, $D\Phi_{F,\,P}(0)$ is given by

$$D\Phi_{F,\,P}(0)u(t) = \big(\dot{u}(t) + g(t)u(t),\, u(0)\big). \tag{2.5.9}$$

One way to see that this is an isomorphism of $C^1_{\{0\}}\big([0,\,\infty)\big)$ onto $C_{\{0\}}\big([0,\,\infty)\big)$ is to notice that if $\dot{u}(t) + g(t)u(t) = 0$, then

$$u(t) = \exp\left(-\int_0^t g(s)\,\mathrm{d}s\right)u(0). \tag{2.5.10}$$

Because $g(t) \geq 1$, this shows that $\big|u(t)\big| \leq e^{-t}\big|u(0)\big|$, and hence that $A(t) = D\Phi_{F,\,P}(0)$ admits an exponential dichotomy with associated projection $P = I$. We then appeal to Corollary 2.3.16 to conclude that $D\Phi_{F,\,P}(0)$ is indeed an isomorphism of $C^1_{\{0\}}\big([0,\,\infty)\big)$ onto $C_{\{0\}}\big([0,\,\infty)\big) \times \mathbb{R}$. In view of Remark 2.4.2, it remains only to check Item 4 in Theorem 2.4.1.

Let $0 \leq s \leq 1$, and suppose that $u \in C^1_{\{0\}}$ solves the system

$$\left.\begin{aligned} \dot{u}(t) + \big((\sin\sqrt{t}) + 2\big)\big(u(t) + u(t)^3\big) = sf(t), \quad t \geq 0 \\ u(0) = s\xi. \end{aligned}\right\} \tag{2.5.11}$$

If u attains its maximum value at $t = 0$, then $\big|u(t)\big| \leq \big|\xi\big|$ for all $t \geq 0$. It then follows from (2.5.11) with $t = 0$ (so $g(t) = 2$) that

$$\big|\dot{u}(t)\big| \leq \big|f(t)\big| + 2(\big|\xi\big| + \big|\xi\big|^3) \tag{2.5.12}$$

for all $t > 0$. Hence, in this case,

$$\left\|u\right\|_{1,\infty} \leq C_{f,\xi} := \left\|f\right\|_{\infty} + 3(\left|\xi\right| + \left|\xi\right|^3). \tag{2.5.13}$$

Otherwise, u attains its maximum absolute value at a point t_0 where $\dot{u}(t_0) = 0$. But then

$$\big((\sin\sqrt{t_0}) + 2\big)\big(u(t_0) + u(t_0)^3\big) = sf(t_0), \tag{2.5.14}$$

from which it follows that

$$\left|u(t_0) + u(t_0)^3\right| \leq \left|f(t_0)\right| \leq \left\|f\right\|_{\infty}. \tag{2.5.15}$$

Since $\left|x + x^3\right| \to \infty$ as $\left|x\right| \to \infty$, this gives an implicit bound

$$\left|u(t_0)\right| \leq C'_f. \tag{2.5.16}$$

Then, just as in (2.5.12) (but with $\left|g(t)\right| \leq 3$ instead of $g(0) = 2$), it follows that

$$\left\|u\right\|_{1,\infty} \leq C''_{f,\xi} := \left\|f\right\|_{\infty} + 4(\left|C'_f\right| + \left|C'_f\right|^3). \tag{2.5.17}$$

Since the constants in estimates (2.5.13) and (2.5.17) depend only on ξ and f, we have verified Item 4 in Theorem 2.4.1.

Having checked all of the hypotheses of Theorem 2.4.1, we conclude that (2.5.2) has a solution $u \in C^1_{\{0\}}$ for every $f \in C_{\{0\}}$ and every $\xi \in \mathbb{R}$.

Remark 2.5.4. Suppose that $p(x)$ in the above example is replaced by $-p(x) = -x - x^3$. Nothing of significance is changed, until we check the Fredholm property. At this point, we find that since $D_xF(t,0)$ is now $-g(t)$, an isomorphism is obtained for $P = 0$ instead of $P = I$. This then results in a slight complication when we derive *a priori* bounds since the bounds can no longer depend on $\left|u(0)\right|$. However, in case $\left|u\right|$ attains its maximum at $t = 0$, it must be that $u(t)^2$ has non-positive derivative. Since this derivative is $2\dot{u}(t)u(t)$, this results in the inequality

$$2\dot{u}(t)u(t) = 4\big(u(0)^2 + u(0)^4\big) + 2sf(0)u(0) \leq 0. \tag{2.5.18}$$

From this inequality it follows that

$$u(0)^2 \leq \left|f(0)\right| \left|u(0)\right|, \tag{2.5.19}$$

from which we obtain a bound that depends only on f for $\left|u(0)\right| = \left\|u\right\|_\infty$. From there, the argument goes through with trivial changes.

As a result, the equation

$$\dot{u}(t) - \big((\sin\sqrt{t}) + 2\big)\big(u(t) + u(t)^3\big) = f(t), \quad t \geq 0 \tag{2.5.20}$$

has a solution $u \in C^1_{\{0\}}\big([0,\infty)\big)$ for all $f \in C_{\{0\}}\big([0,\infty)\big)$. $\Diamond$

2.5.2 A gradient example

Now we examine a more general situation. Let $d \in \mathbb{N}$, let g be a bounded, uniformly continuous map from $[0, \infty)$ into $\mathbb{R}^{d\times d}$, and let $\phi \in C^2(\mathbb{R}^d, \mathbb{R})$. Let

$$F(t, x) := g(t)\nabla\phi(x). \tag{2.5.21}$$

We assume that for some $\alpha > 0$, g satisfies the quadratic form estimate

$$\big\langle g(t)x, x\big\rangle \geq \alpha \left|x\right|^2, \quad \forall x \in \mathbb{R}^d, \forall t \geq 0. \tag{2.5.22}$$

We suppose that ϕ satisfies the following four conditions. First,

$$\nabla\phi(x) = 0 \iff x = 0. \tag{2.5.23}$$

Second, at $x = 0$ the Hessian $\nabla^2\phi(0) \in \mathbb{R}^{d\times d}$ satisfies

$$\det \nabla^2\phi(0) \neq 0. \tag{2.5.24}$$

Third, ϕ satisfies one of the two growth conditions

$$\lim_{x \to\infty} \phi(x) = \infty; \tag{2.5.25a}$$

OR

$$\lim_{x \to\infty} \phi(x) = -\infty. \tag{2.5.25b}$$

Finally, $\nabla\phi$ satisfies the growth condition

$$\lim_{x \to\infty} \left|\nabla\phi(x)\right| = \infty. \tag{2.5.26}$$

In addition, it will be no loss of generality to assume

$$\phi(0) = 0; \tag{2.5.27}$$

we just replace ϕ by $\phi - \phi(0)$. This has no bearing on any of these conditions, and in no way changes the function F.

Remark 2.5.5. For illustrative purposes, we will think of ϕ as the altitude function of a landscape. It will be helpful to interpret $\nabla\phi$ as pointing out the direction of steepest ascent. Hence, if condition (2.5.25a) holds, the landscape has a single valley at $x = 0$, no peaks, and slopes upward at an ever greater rate at ever greater distances from the origin. The situation is reversed if condition (2.5.25b) holds. Accordingly, we will see that the Cauchy problem is well-posed in the former case. In the latter case the problem with no initial conditions can be solved. $\Diamond$

We now begin to check the hypotheses of Theorem 2.4.1. That F satisfies conditions (2.1.1a)–(2.1.1b) on page 16 is obvious, and so Item 1 of Theorem 2.4.1 is satisfied. We move on to the Fredholm property. Set

$$H := \nabla^2\phi(0). \tag{2.5.28}$$

Thus, if we set $A(t) := D_x F(t,\, 0)$, then

$$A(t) = g(t)H. \tag{2.5.29}$$

We will now show that A admits an exponential dichotomy with projection either $\Pi = 0$ or $\Pi = I$ if ϕ satisfies growth condition (2.5.25a) or (2.5.25b), respectively. Let $x \in \mathbb{R}^d$ be given. Recall that H^{-1} exists, by (2.5.24). Notice that H is symmetric, because ϕ is C^2. Using this with (2.5.22), we find that for all $t \geq 0$,

$$\left\langle HA(t)x,\, x\right\rangle = \left\langle Hg(t)Hx,\, x\right\rangle \tag{2.5.30}$$

$$= \left\langle g(t)Hx,\, Hx\right\rangle \tag{2.5.31}$$

$$\geq \alpha \left|H(x)\right| \tag{2.5.32}$$

$$\geq \frac{\alpha}{\left|H^{-1}\right|}\left|x\right|. \tag{2.5.33}$$

This implies that as a quadratic form (even on $\mathbb{C}^{d\times d}$), the symmetric part of $HA(t)$ is positive:

$$HA(t) + A(t)^*H \geq 2\frac{\alpha}{\left|H^{-1}\right|}I. \tag{2.5.34}$$

Since H is independent of t, we conclude from Lemma 2.3.27 that $A(t) = g(t)H$ admits an exponential dichotomy on $[0,\, \infty)$, with *some* projection Π.

To find Π, suppose that $u \in C^1$ satisfies the equation $\dot{u}(t) + g(t)Hu(t) = 0$ and consider the function

$$\psi(t) := \left\langle Hu(t),\, u(t)\right\rangle. \tag{2.5.35}$$

Notice that for all $t \geq 0$,

$$\dot{\psi}(t) = \left\langle H\dot{u}(t),\, u(t)\right\rangle + \left\langle Hu(t),\, \dot{u}(t)\right\rangle \tag{2.5.36}$$

$$= 2\left\langle -g(t)Hu(t),\, Hu(t)\right\rangle \tag{2.5.37}$$

$$\leq -2\alpha\left|Hu(t)\right|^2, \tag{2.5.38}$$

according to (2.5.22). Therefore, ψ is a non-increasing function of t. Suppose for the moment that it is condition (2.5.25a) that holds of ϕ, so that $\phi(x) \to \infty$ as $\left|x\right| \to \infty$. Then $\phi(0) = 0$

is the absolute *minimum* value of ϕ, whence the eigenvalues of $H = \nabla^2\phi(0)$ are all positive. Hence, if $\lambda_{\min}$ is the least of these, then

$$\langle Hx,\, x\rangle \geq \lambda_{\min} \left|x\right|^2, \quad \forall x \in \mathbb{R}^d. \tag{2.5.39}$$

In this case, (2.5.36)–(2.5.38) show that $\left|u(t)\right|^2$, and hence $\left|u(t)\right|$, are bounded above by non-increasing functions of t. Recall the growth condition (2.3.36) in Corollary 2.3.11 on page 60. This growth implies that $I - \Pi$ must be zero, showing that $\Pi = I$.

On the other hand, if condition (2.5.25b) holds of ϕ, then $\phi(0) = 0$ is the absolute *maximum* value of ϕ, whence the eigenvalues of $H = \nabla^2\phi(0)$ are all negative. As a result, one finds that $\left|u(t)\right|$ is instead bounded below by a nondecreasing function of t, which implies that $\Pi = 0$. We record these results for use throughout this example:

Lemma 2.5.6. *In the context of the present example, If condition (2.5.25a) holds, then $A(t) = D_xF(t,\, x) = g(t)\nabla^2\phi(0)$ admits an exponential dichotomy with projection $\Pi = I$, whence $D\Phi_{F,\,I}$ is an isomorphism of $C^1_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big)$ onto $C_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big) \times \mathbb{R}^d$.*

If condition (2.5.25b) holds, then A admits an exponential dichotomy with projection $\Pi = 0$, whence $D\Phi_{F,\,0}$ is an isomorphism of $C^1_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big)$ onto $C_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big) \times \{0\}$.

According to Remark 2.4.2 on page 87, this settles both Items 3 and 5 in Theorem 2.4.1.

We now check the admissibility of $\omega(F)$. Suppose that $G \in \omega(F)$. According to Lemma 2.2.23, G is of the form $G(t,\, x) = B(t)\nabla p(x)$ for some $B \in \omega(g)$. We claim that

$$\langle B(t)x,\, x\rangle \geq \alpha \left|x\right|^2, \quad \forall x \in \mathbb{R}^d,\, \forall t \in \mathbb{R}. \tag{2.5.40}$$

Indeed, there is a sequence $(\sigma_n) \subset [0,\,\infty)$ such that $\sigma_n \to \infty$ and $g(t + \sigma_n) \to B(t)$ for each t (even uniformly on compact sets). For each t and sufficiently large n, we have $t + \sigma_n \geq 0$.

For all such n, inequality (2.5.22) becomes

$$\big\langle g(t+\sigma_n)x,\, x\big\rangle \geq \alpha \left|x\right|^2, \quad \forall x \in \mathbb{R}^d. \tag{2.5.41}$$

Letting $n \to \infty$ yields inequality (2.5.40). One immediate implication is that $G(t,\, x) = 0$ if and only if $\nabla p(x) = 0$, which happens only for $x = 0$. Since $G \in \omega(F)$ is arbitrary, this means that $\omega_0(F) = \{0\}$, which is totally disconnected. To finish checking that $\omega(F)$ is admissible, let $u \in C^1$ be a solution of the equation

$$\dot{u}(t) + B(t)\nabla p(x) = 0 \tag{2.5.42}$$

that is bounded on $\mathbb{R}$. Consider the function

$$\psi(t) := \phi\big(u(t)\big). \tag{2.5.43}$$

We have

$$\dot{\psi}(t) = \big\langle \nabla\phi\big(u(t)\big),\, \dot{u}(t)\big\rangle \tag{2.5.44}$$

$$= \big\langle \nabla\phi\big(u(t)\big),\, -B(t)\nabla\phi\big(u(t)\big)\big\rangle \tag{2.5.45}$$

$$\leq -\alpha \left|\nabla\phi\big(u(t)\big)\right|^2 \tag{2.5.46}$$

Assuming that u is not the zero solution,[14] $\dot{\psi}(t)$ is *strictly* negative (see (2.5.23)), so that $\psi(t) = \phi\big(u(t)\big)$ is strictly decreasing in t. Suppose for the moment that condition (2.5.25a) holds of ϕ. Since $x = 0$ is the only critical point of ϕ and $\phi(x) \to \infty$ as $\left|x\right| \to \infty$, $\phi(0) = 0$ is the absolute minimum value of ϕ. Since $u(0) > 0$ and $\phi\big(u(t)\big)$ is strictly decreasing in t,

$$\phi\big(u(t)\big) > \phi\big(u(0)\big) > 0, \quad \forall t < 0. \tag{2.5.47}$$

[14] Recall that we are not concerned with constant solutions to equation (2.5.42).

This implies that there is some $\epsilon > 0$ such that $\left|u(t)\right| > \epsilon$ for all $t < 0$. In turn, this implies that $\left|\nabla\phi\big(u(t)\big)\right|$ is also bounded away from zero uniformly in $t < 0$. With inequality (2.5.46), this implies that $\psi(t) \to \infty$ as $t \to -\infty$. This can only happen if $u(t)$ is unbounded as $t \to -\infty$. Similarly, if condition (2.5.25b) holds of ϕ, we find that $u(t)$ is unbounded as $t \to \infty$. We conclude that $\omega(F)$ is admissible.

Remark 2.5.7. Had we not assumed that $\nabla\phi(x) = 0$ if and only if $x = 0$, it would have remained possible that u is a heteroclinic orbit connecting two equilibrium solutions of (2.5.42). It seems unlikely that this possibility could be ruled out. $\diamondsuit$

To obtain an existence result, it remains only to settle Item 4 in Theorem 2.4.1 on page 83. We suppose that $u \in C^1_{\{0\}}$ solves the differential equation

$$\dot{u}(t) + g(t)\nabla\phi\big(u(t)\big) = sf(t) \quad \forall t \geq 0, \tag{2.5.48}$$

where $f \in C_{\{0\}}$ and $0 \leq s \leq 1$. (We do not yet specify an initial condition.) We turn once again to the function ψ defined by

$$\psi(t) := \phi\big(u(t)\big); \tag{2.5.49}$$

in view of equation (2.5.48), one has

$$\dot{\psi}(t) = \left\langle sf(t) - g(t)\nabla\phi\big(u(t)\big),\, \nabla\phi\big(u(t)\big)\right\rangle. \tag{2.5.50}$$

From here, the argument differs slightly depending on whether $\phi(x) \to \infty$ as $\left|x\right| \to \infty$ (condition (2.5.25a)) or $\phi(x) \to -\infty$ as $\left|x\right| \to \infty$ (condition (2.5.25b)). Suppose first that we are in the latter situation, so that Lemma 2.5.6 dictates that we choose $P = 0$. Thus, we add no initial condition to equation (2.5.48). Also, $\phi(x) \leq 0$ for all $x \in \mathbb{R}^d$. Thus, if the maximum value of $\left|\psi(t)\right| = -\psi(t)$ is attained at a point $t_0 \geq 0$, then $\dot{\psi}(t_0) \leq 0$, even if $t_0 = 0$. In this case, equation (2.5.50) implies that

$$\left\langle sf(t_0),\, \nabla\phi\big(u(t_0)\big)\right\rangle \geq \left\langle g(t_0)\nabla\phi\big(u(t_0)\big),\, \nabla\phi\big(u(t_0)\big)\right\rangle . \tag{2.5.51}$$

With use of (2.5.22), this implies that

$$\left\langle \nabla\phi\big(u(t_0)\big),\, sf(t_0)\right\rangle \geq \alpha \left|\nabla\phi\big(u(t_0)\big)\right|^2 , \tag{2.5.52}$$

so that

$$\left|\nabla\phi\big(u(t_0)\big)\right| \leq \alpha^{-1} \left|sf(t_0)\right| \leq \alpha^{-1} \left\|f\right\|_\infty . \tag{2.5.53}$$

Since $\left|\nabla\phi(x)\right| \to \infty$ as $\left|x\right| \to \infty$, we have proved that there is $C_1 = C_1(f)$ such that if the maximum value of $\left|\psi\right|(t)$ on $[0,\, \infty)$ occurs at $t = t_0$, then $\left|u(t_0)\right| \leq C_1$. This, in turn, gives a bound for $\left|\psi(t_0)\right|$ and hence for $\left|\psi(t)\right| = \left|\phi\big(u(t)\big)\right|$ on $[0,\, \infty)$:

$$\max_{t\geq 0} \left|\phi\big(u(t)\big)\right| \leq \max_{\left|x\right| \leq C_1} \left|\phi(x)\right| =: C_2 = C_2(f). \tag{2.5.54}$$

Since $\left|\phi(x)\right| \to \infty$ as $\left|x\right| \to \infty$, this proves the existence of the desired bound for $\left|u(t)\right|$ on $[0,\, \infty)$. We have found a bound for $\left\|u\right\|_\infty$ that depends only on f.

By virtue of (2.5.48), this is sufficient, for now

$$\left|\dot{u}(t)\right| \leq \left|g(t)\nabla\phi\big(u(t)\big)\right| + \left|f(t)\right| \tag{2.5.55}$$

provides a bound for $\left\|\dot{u}\right\|_\infty$, and therefore also for $\left\|u\right\|_{1,\infty}$. We have finished checking the hypotheses of Theorem 2.4.1 in the case that $\phi(x) \to -\infty$ as $\left|x\right| \to \infty$, so we state the result before moving on.

Example 2.5.8. In conclusion, if g and ϕ satisfy all of the conditions (2.5.22) through (2.5.27), with the exception of (2.5.25a), then there is a solution $u \in C^1_{\{0\}}\big([0,\infty),\mathbb{R}^d\big)$ to the equation

$$\dot{u}(t) + g(t)\nabla\phi\big(u(t)\big) = f(t) \quad \forall t \geq 0 \tag{2.5.56}$$

for each given $f \in C_{\{0\}}\big([0,\infty),\mathbb{R}^d\big)$. ♦

We now return to the case that ϕ satisfies (2.5.25a). Thus, $\phi(x) \to \infty$ as $|x| \to \infty$, and $\phi \geq 0$. Also, Lemma 2.5.6 dictates that we choose $P = I$. We have found to this point that $\Phi_{F,I}$ satisfies Items 1, 2, 3, and 5 in Theorem 2.4.1. Given $f \in C_{\{0\}}$, $\xi \in \operatorname{rge} I = \mathbb{R}^d$, and $0 \leq s \leq 1$, it remains to find *a priori* bounds for solutions $u \in C^1_{\{0\}}$ to the Cauchy problem

$$\left.\begin{aligned} \dot{u}(t) + g(t)\nabla\phi\big(u(t)\big) = sf(t), \qquad \forall t \geq 0 \\ u(0) = s\xi. \end{aligned}\right\} \tag{2.5.57}$$

Let $u \in C^1_{\{0\}}$ be such a solution. Once again, take

$$\psi(t) := \phi\big(u(t)\big); \tag{2.5.58}$$

we still have

$$\dot{\psi}(t) = \big\langle sf(t) - g(t)\nabla\phi\big(u(t)\big),\, \nabla\phi\big(u(t)\big)\big\rangle\,. \tag{2.5.59}$$

If the maximum of $|\psi(t)|$ is attained at a strictly positive value $t = t_0 > 0$, then $\dot{\psi}(t_0) = 0$, and the technique leading up to Example 2.5.8 (which only used $\dot{\psi}(t_0) \geq 0$) provides the desired bound. This does not work if the maximum value of $|\psi|$ is attained only at $t = 0$, for then $\dot{\psi}(t_0) \leq 0$ since ψ is non-negative. However, $u(t_0) = s\xi$, so $|\psi(0)| = |\phi(s\xi)|$. This maximum value is then bounded by some $C_1 = C_1(\xi) := \max_{|x|\leq|\xi|}|\phi(x)|$. We then have

$$C_1 \geq |\psi(t)| = \big|\phi\big(u(t)\big)\big| \tag{2.5.60}$$

for all $t \geq 0$. We use once more the assumption that $\phi(x) \to \infty$ as $\left|x\right| \to \infty$ to conclude that there exists some $C_2 > 0$ such that $\left|u(t)\right| \leq C_2$ for all $t \geq 0$. Once again, we use the equation for $\dot{u}$ in (2.5.57) to obtain a bound for $\left\|u\right\|_{1,\infty}$. We summarize:

Example 2.5.9. In conclusion, if g and ϕ satisfy all of the conditions (2.5.22) through (2.5.27), with the exception of (2.5.25b), then there is a solution $u \in C^1_{\{0\}}\big([0,\infty), \mathbb{R}^d\big)$ to the initial value problem

$$\left.\begin{aligned} \dot{u}(t) + g(t)\nabla\phi\big(u(t)\big) = f(t) \qquad &\forall t \geq 0 \\ u(0) = \xi \qquad\qquad & \end{aligned}\right\} \tag{2.5.61}$$

for each given $f \in C_{\{0\}}\big([0,\infty), \mathbb{R}^d\big)$ and $\xi \in \mathbb{R}^d$. ♦

Remark 2.5.10. A similar class of examples is based on the assumption that for some $\alpha > 0$, all $t \geq 0$ and all $x \in \mathbb{R}^d$ that

$$\big\langle F(t, x), x\big\rangle \geq \alpha \left|x\right|^2. \tag{2.5.62}$$

This assumption is neither more nor less general than (2.5.22), even if $F(t, x) = g(t)\nabla\phi(x)$. To sketch how this example proceeds, assume that Item 1 of Theorem 2.4.1 on page 83 holds. Assume also that $D\Phi_{F,I}(0)$ is an isomorphism of $C^1_{\{0\}}$ onto $C_{\{0\}} \times \mathbb{R}^d$, and that $F(t, x)$ is odd in x, so that Items 3 and 5 are satisfied, and so that we are working with the Cauchy problem. To handle properness, first note that (2.5.62) is inherited by all $G \in \omega(F)$, for all $t \in \mathbb{R}$. (Otherwise, $\big\langle G(t, x), x\big\rangle < \alpha\left|x\right|^2$. But $G(t, x) = \lim_{n\to\infty} F(t + \sigma_n, x)$.) This shows that $\omega_0(F)$ is the totally disconnected set $\{0\}$. Next, if $\dot{u}(t) + G\big(t, u(t)\big) = 0$, then $\dot{\psi}(t) \leq -\alpha\left|u(t)\right|^2$ for $\psi(t) := \left|u(t)\right|^2$. This shows that if u is not the zero solution, then $\left|u(t)\right| \to \infty$ as $t \to -\infty$.

For *a priori* bounds, we take a similar approach. If $\dot{u}(t) + F\big(t,\, u(t)\big) = sf(t)$, then at an interior maximum of ψ, we have $\dot{\psi}(t_0) = 0$ so that

$$\big\langle sf(t_0),\, u(t_0)\big\rangle = \big\langle F\big(t_0,\, u(t_0)\big),\, u(t_0)\big\rangle \geq \alpha \left|u(t_0)\right|^2. \tag{2.5.63}$$

This, along with $\left|u(0)\right| = \left|s\xi\right|$, gives bounds that depend only on f and ξ.

A similar example (but with $P = 0$ and no initial condition) results if (2.5.62) is replaced by

$$\big\langle F(t,\, x),\, x\big\rangle \leq -\alpha \left|x\right|^2. \tag{2.5.64}$$

2.5.3 A second-order example

We now consider a class of second-order problems, with either Dirichlet or Neumann conditions at $t = 0$. The Dirichlet problem is of the form

$$\left.\begin{aligned} \ddot{v}(t) + G\big(t,\, v(t)\big) = g(t) \qquad &\forall t \geq 0;\\ v(0) = \xi. \qquad & \end{aligned}\right\} \tag{2.5.65}$$

The Neumann problem is of the form

$$\left.\begin{aligned} \ddot{v}(t) + G\big(t,\, v(t)\big) = g(t) \qquad &\forall t \geq 0;\\ \dot{v}(0) = \xi. \qquad & \end{aligned}\right\} \tag{2.5.66}$$

We seek solutions $v \in C^2_{\{0\}}\big([0,\, \infty),\, \mathbb{R}^d\big)$ to these problems, meaning in particular that we require $v(t)$, $\dot{v}(t)$, and also $\ddot{v}(t)$ to tend to zero as $t \to \infty$. We stipulate that the following conditions hold of $G\colon [0,\, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$. First, G satisfies the three conditions (2.1.1a), (2.1.1b), and (2.1.1c) on page 16, (with "F" replaced by "G"). Second, we assume that for each t, the function $G(t,\,\cdot)$ is odd, *i.e.* $G(t,\,-y) = -G(t,\,y)$. We assume that there is some $\alpha > 0$ such that

$$\left\langle G(t,\,y),\,y\right\rangle \leq -\alpha\left|y\right|^2, \quad \forall x \in \mathbb{R}^d,\, \forall t \geq 0. \tag{2.5.67}$$

Put $B(t) := D_yG(t,\,0)$. We assume that B is asymptotically constant.[15] We will also require that the eigenvalues of $B(t)$ eventually stay away from the positive real axis. To state this requirement precisely, we define the following open region in the complex plane for each $\epsilon > 0$:

$$R_\epsilon := \left\{-z^2\colon z = a+bi,\, a\in\mathbb{R},\, b\in\mathbb{R},\left|a\right| > \epsilon\right\}. \tag{2.5.68}$$

Notice that $[0,\,\infty)$ is in the interior of $\mathbb{C}\setminus R_\epsilon$, and that $\bigcap_{\epsilon>0}(\mathbb{C}\setminus R_\epsilon) = [0,\,\infty)$. We require that there be $\epsilon > 0$ and $T \geq 0$ such that for all $t \geq T$, all eigenvalues of $B(t)$ lie in R_ϵ. This requirement is met, for example, if $B(t)$ is uniformly strictly negative definite for $t \geq T$.

Next, we formulate the usual equivalent first order problem. We identify each $x \in \mathbb{R}^{2d}$ with a column vector

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x_1,\, x_2 \in \mathbb{R}^d \tag{2.5.69}$$

in the canonical way. We then define $F\colon [0,\,\infty)\times\mathbb{R}^{2d}\to\mathbb{R}^{2d}$ by

$$F(t,\,x) := \begin{bmatrix} -x_2 \\ G(t,\,x_1) \end{bmatrix}, \tag{2.5.70}$$

and $f\colon [0,\,\infty)\to\mathbb{R}^{2d}$ by

$$f(t) := \begin{bmatrix} 0 \\ g(t) \end{bmatrix}. \tag{2.5.71}$$

[15] See Example 2.2.24 on page 48, as well as Remark 2.2.25 that follows. Of course, this covers the case that $\lim_{t\to\infty} B(t)$ exists.

We bring in the projections

$$P_1 x := \begin{bmatrix} x_1 \\ \\ 0 \end{bmatrix}, \text{ and} \qquad\qquad P_2 x := \begin{bmatrix} 0 \\ \\ x_2 \end{bmatrix} \tag{2.5.72}$$

It is then easy to check that $u = \left[\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right] \in C^1_{\{0\}}\big([0, \infty), \mathbb{R}^{2d}\big)$ solves

$$\left.\begin{aligned} \dot{u}(t) + F\big(t, u(t)\big) = f(t) \qquad & \forall t \geq 0; \\ P_i u(0) = \xi \qquad & \end{aligned}\right\} \tag{2.5.73}$$

if and only if $v = u_1 \in C^2_{\{0\}}\big([0, \infty), \mathbb{R}^d\big)$ solves the Dirichlet problem (2.5.65) (if $i = 1$) or the Neumann problem (2.5.66) (if $i = 2$).

We will prove existence by application of Theorem 2.4.1. First, it is clear that F satisfies conditions (2.1.1a), (2.1.1c), and (2.1.1b) on page 16, because G satisfies the same. Next, we recall that $B(t) := D_y G(t, 0)$ and note that the standard block matrix representation for $A(t) = D_x F(t, 0)$ is

$$A(t) := \begin{bmatrix} 0 & -I \\ \\ B(t) & 0 \end{bmatrix}. \tag{2.5.74}$$

Thus, A is asymptotically autonomous because B is assumed to be asymptotically autonomous. We are in the situation of Example 2.3.33 on page 81, so we examine $\omega(A)$. First, it follows immediately from (2.5.74) that $T \in \omega(A)$ if and only if $T = \left[\begin{smallmatrix} 0 & -I \\ S & 0 \end{smallmatrix}\right]$ for some $S \in \omega(B)$. Because $A - \lambda I$ is a block matrix whose blocks commute, an easy calculation shows that

$$\det(A - \lambda I_{2d}) = \det(S + \lambda^2 I_d) \tag{2.5.75}$$

as polynomials in λ. Thus, if the eigenvalues of S are $\sigma(S) = \{\mu_1, \mu_2, \ldots, \mu_d\}$ (the list reflecting the algebraic multiplicities), then the eigenvalues of T are (for any branch of $\sqrt{\cdot}$)

$$\sigma(T) = \{\pm\sqrt{\mu_1}, \pm\sqrt{\mu_2}, \ldots, \pm\sqrt{\mu_d}\}. \tag{2.5.76}$$

Note that as verified in Remark 2.2.25 on page 49,

$$\omega(B) = \bigcap_{n\in\mathbb{N}} \overline{B\big([n,\, \infty)\big)}. \tag{2.5.77}$$

Since $\sigma(B)$ is always contained in R_ϵ, this shows that $\sigma(S)$ is contained in $\overline{R_\epsilon} \subset R_{\epsilon/2}$. According to the definition of $R_{\epsilon/2}$, and because of (2.5.76), this shows that each of the d pairs $\pm\sqrt{\mu_j}$ of eigenvalues of T consists of one whose real part is greater than $\epsilon/2$, and one whose real part is less than $\epsilon/2$. In summary, all of the eigenvalues of T are of a distance at least $\epsilon/2$ from the imaginary axis, and exactly half of these (in algebraic count) have positive real part. Since $\operatorname{rank} P_i = d$, for $i = 1, 2$, we conclude from Example 2.3.33 that the operator Φ_{F,P_i} is Fredholm of index zero from $C^1_{\{0\}}\big([0,\, \infty),\, \mathbb{R}^{2d}\big)$ into $C_{\{0\}}\big([0,\, \infty),\, \mathbb{R}^{2d}\big) \times \operatorname{rge} P_i$, for $i = 1, 2$.

Because $G(t,\, y)$ is assumed to be odd in y, it is clear that $F(t,\, x)$ is odd in x. Thus, by Borsuk's Theorem (see Remark 2.4.3 on page 87), it remains only to verify Items 2 and 4 in Theorem 2.4.1 on page 83. Notice that because of inequality (2.5.67), all of the functions $K \in \omega(G)$ satisfy the same inequality for all $t \in \mathbb{R}$:

$$\big\langle K(t,\, y),\, y\big\rangle \le -\alpha \left|y\right|^2, \quad \forall x \in \mathbb{R}^d,\, \forall t \in \mathbb{R}. \tag{2.5.78}$$

One implication is that $\omega_0(G) = \{0\} \subset \mathbb{R}^d$. It follows immediately from the definition (2.5.70) of F that each $H \in \omega(F)$ is of the form

$$H(t,\, x) := \begin{bmatrix} -x_2 \\ K(t,\, x_1) \end{bmatrix} \tag{2.5.79}$$

for some (unique to H) $K \in \omega(G)$. For one thing, this shows that $\omega_0(F) = \{0\} \subset \mathbb{R}^{2d}$, which is totally disconnected. Now let $H \in \omega(F)$ be given, and suppose that $u \in C^1\big(\mathbb{R},\, \mathbb{R}^{2d}\big)$ is a C^1 solution to

$$\dot{u}(t) + H\big(t,\, u(t)\big) = 0. \tag{2.5.80}$$

Then $v = u_1 \in C^2\big(\mathbb{R},\, \mathbb{R}^d\big)$ is a solution to

$$\ddot{v}(t) + K\big(t,\, v(t)\big) = 0, \tag{2.5.81}$$

where $K \in \omega(G)$. We are to show that if u is bounded on $\mathbb{R}$, then u must be constant. We introduce the function

$$\psi(t) := \frac{1}{2}\left|v(t)\right|^2 = \frac{1}{2}\left\langle v(t),\, v(t)\right\rangle. \tag{2.5.82}$$

We calculate

$$\dot{\psi}(t) = \left\langle \dot{v}(t),\, v(t)\right\rangle, \tag{2.5.83}$$

and so

$$\ddot{\psi}(t) = \left\langle \dot{v}(t),\, \dot{v}(t)\right\rangle + \left\langle \ddot{v}(t),\, v(t)\right\rangle \tag{2.5.84}$$

$$= \left|\dot{v}(t)\right|^2 - \left\langle G\big(t,\, v(t)\big),\, v(t)\right\rangle \tag{2.5.85}$$

$$\geq \alpha \left|v(t)\right|^2. \tag{2.5.86}$$

We see that ψ is a convex function. As a result, if there is any point t_0 where $\dot{\psi}(t_0) > 0$, then $\psi(t) \to \infty$ as $t \to \infty$. Likewise, if $\dot{\psi}(t_0) < 0$ for some t_0, then $\psi(t) \to \infty$ as $t \to -\infty$. Therefore, if ψ is bounded on $\mathbb{R}$, then $\dot{\psi} = 0$, so that ψ is constant.

If u is bounded on $\mathbb{R}$, then $v = u_1$ is bounded, so that $\psi = \left|v\right|^2/2$ is bounded and thus constant, by the preceding argument. But inequalities (2.5.84)–(2.5.86) show that this is possible only if $v(t) = 0$; otherwise ψ is strictly convex. Thus, $v = 0$ is constant. Since

$u_1 = v$ and $u_2 = \dot{v}$, u is constant, as desired. We have proved that $\omega(F)$ is admissible, which takes care of Item 2 in Theorem 2.4.1 on page 83. It remains only to find *a priori* bounds, as per Item 4.

We consider first $i = 1$ in (2.5.73), which corresponds to the Dirichlet problem (2.5.65). Let $g \in C_{\{0\}}\big([0, \infty), \mathbb{R}^d\big)$, $\xi \in \mathbb{R}^d$, and $0 \le s \le 1$. Put $f = \left[\begin{smallmatrix} 0 \\ g \end{smallmatrix}\right]$, and suppose that $u = \left[\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right] \in C^1_{\{0\}}\big([0, \infty), \mathbb{R}^{2d}\big)$ solves

$$\left.\begin{aligned} \dot{u}(t) + F\big(t, u(t)\big) &= sf(t) \qquad \forall t \ge 0; \\ P_i u(0) &= \left[\begin{smallmatrix} s\xi \\ 0 \end{smallmatrix}\right]. \end{aligned}\right\} \tag{2.5.87}$$

We seek bounds for $\left\|u\right\|_{1,\infty}$ that depend only on f and ξ. We find these bounds by returning to the second-order formulation. Taking $v = u_1$, we have

$$\left.\begin{aligned} \ddot{v}(t) + G\big(t, v(t)\big) &= sg(t) \qquad \forall t \ge 0; \\ v(0) &= s\xi. \end{aligned}\right\} \tag{2.5.88}$$

Once again we use the function

$$\psi(t) := \frac{1}{2}\left|v(t)\right|^2 = \frac{1}{2}\left\langle v(t), v(t)\right\rangle. \tag{2.5.89}$$

Again,

$$\dot{\psi}(t) = \left\langle \dot{v}(t), v(t)\right\rangle, \tag{2.5.90}$$

and so this time

$$\ddot{\psi}(t) = \left\langle \dot{v}(t), \dot{v}(t)\right\rangle + \left\langle \ddot{v}(t), v(t)\right\rangle \tag{2.5.91}$$

$$= \left|\dot{v}(t)\right|^2 + \left\langle sg(t) - G\big(t, v(t)\big), v(t)\right\rangle. \tag{2.5.92}$$

Because ψ is continuous and $\psi(t) \to 0$ as $t \to \infty$, the maximum value of $\psi(t)$ is attained at some $t_0 \geq 0$. If $t_0 = 0$, then

$$\max_{t\geq 0} \psi(t) = \psi(0) = \frac{1}{2} \left|s\xi\right|^2, \tag{2.5.93}$$

so that $\left|v(t)\right| \leq \left|\xi\right|$ for all $t \geq 0$. Otherwise, the maximum value of $\psi(t)$ is attained at an interior point t_0 of $[0, \infty)$, so that $\ddot{\psi}(t_0) \leq 0$. With (2.5.91)–(2.5.92), this implies that

$$-\left\langle G\big(t_0, v(t_0)\big), v(t_0)\right\rangle \leq -\left\langle sg(t_0), v(t_0)\right\rangle. \tag{2.5.94}$$

Because G satisfies (2.5.67), this implies that $\alpha \left|v(t_0)\right|^2 \leq -s \left\langle g(t_0), v(t_0\right\rangle$, and hence that

$$\left|v(t_0)\right| \leq \alpha^{-1} \left\|g\right\|_\infty. \tag{2.5.95}$$

Thus, whether $t_0 = 0$ or $t_0 > 0$, there is $C_1 = C_1(g, \xi)$ such that

$$\left\|v\right\|_\infty \leq C_1 \tag{2.5.96}$$

This immediately gives a bound for the second derivative $\ddot{v}$. Because $\ddot{v} = -G(t, v)$, we have

$$\left\|\ddot{v}\right\|_\infty \leq C_2 := \max_{\substack{|y| \leq C_1 \\ t\geq 0}} \left|G(t, y)\right|. \tag{2.5.97}$$

Notice that $C_2 < \infty$ because G is assumed to satisfy condition (2.1.1b) on page 16. This control over both v and $\ddot{v}$ gives control over $\dot{v}$ as follows. Let $t \geq 0$ be given; we have

$$\begin{aligned} v(t+1) - v(t) &= \int_t^{t+1} \dot{v}(s)\,\mathrm{d}s & (2.5.98)\\ &= \int_t^{t+1} \dot{v}(t)\,\mathrm{d}s + \int_t^{t+1} \dot{v}(s) - \dot{v}(t)\,\mathrm{d}s & (2.5.99)\\ &= \dot{v}(t) + \int_t^{t+1} \dot{v}(s) - \dot{v}(t)\,\mathrm{d}s. & (2.5.100)\end{aligned}$$

Hence,

$$\begin{aligned}
\left|\dot v(t)\right| &\le \left|v(t+1) - v(t)\right| + \int_t^{t+1} \left|\dot v(s) - \dot v(t)\right| \mathrm{d}s && (2.5.101)\\
&\le 2C_1 + \int_t^{t+1} C_2(s-t)\,\mathrm{d}s && (2.5.102)\\
&= 2C_1 + C_2/2. && (2.5.103)
\end{aligned}$$

This is the desired bound for $\dot v$. Finally, having a bound for v, $\dot v$ and $\ddot v$ is equivalent to having a bound for $u = [v,\, \dot v]^T$ and $\dot u = [\dot v,\, \ddot v]^T$. Having verified all of the hypotheses of Theorem 2.4.1 on page 83, we conclude that there is a solution for (2.5.87) when $i = 1$, and hence for the Dirichlet problem (2.5.65). We record this result before completing the Neumann problem.

Example 2.5.11. Suppose that G satisfies all of the stated conditions, starting on page 102 and ending just before (2.5.69). Then for each $g \in C_{\{0\}}\big([0,\, \infty),\, \mathbb{R}^d\big)$ and $\xi \in \mathbb{R}^d$, the Dirichlet problem

$$\left.\begin{aligned}
\ddot v(t) + G\big(t,\, v(t)\big) &= g(t) \qquad \forall t \ge 0;\\
v(0) &= \xi
\end{aligned}\right\} \qquad (2.5.104)$$

has a solution $v \in C^2\big([0,\, \infty),\, \mathbb{R}^d\big)$ such that $\left|v(t)\right|$, $\left|\dot v(t)\right|$, and $\left|\ddot v(t)\right|$ all tend to 0 as $t \to \infty$.

♦

We now complete the study of the Neumann problem. It remains only to find *a priori* bounds. Hence, we take $i = 2$, $g \in C_{\{0\}}\big([0,\, \infty),\, \mathbb{R}^d\big)$, $\xi \in \mathbb{R}^d$, and $0 \le s \le 1$. Put $f = \left[\begin{smallmatrix} 0 \\ g \end{smallmatrix}\right]$, and suppose that $u = \left[\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right] \in C^1_{\{0\}}\big([0,\, \infty),\, \mathbb{R}^{2d}\big)$ solves

$$\left.\begin{aligned}
\dot u(t) + F\big(t,\, u(t)\big) &= sf(t) \qquad \forall t \ge 0;\\
P_i u(0) &= \left[\begin{smallmatrix} 0 \\ s\xi \end{smallmatrix}\right].
\end{aligned}\right\} \qquad (2.5.105)$$

It is clear that the bounds found in the Dirichlet problem are valid for any solution v to

$$\left.\begin{aligned} \ddot{v}(t) + G\big(t,\, v(t)\big) &= sg(t) \qquad \forall t \geq 0; \\ \dot{v}(0) &= s\xi \end{aligned}\right\} \tag{2.5.106}$$

such that $\psi(t) = (1/2)\left|v(t)\right|^2$ attains its maximum at some $t_0 > 0$, since these bounds did not depend in the initial conditions. We need only find a bound for $\left|v(0)\right|$ under the assumptions that $\psi(t)$ attains its maximum at $t = 0$. Consider that for all $t \geq 0$, we have

$$\ddot{\psi}(t) = \left|\dot{v}(t)\right|^2 + \big\langle sg(t) - G\big(t,\, v(t)\big),\, v(t)\big\rangle \tag{2.5.107}$$

$$\geq \alpha \left|v(t)\right|^2 - \left\|g\right\|_\infty \left|v(t)\right| \tag{2.5.108}$$

$$= \alpha\psi(t) - \left\|g\right\|_\infty \sqrt{\psi(t)}. \tag{2.5.109}$$

Since $\alpha > 0$ there is $M > 0$ such that if $\psi(t) \geq M$, then $\ddot{\psi}(t) \geq (\alpha/2)\psi(t)$. Therefore, if $[0,\, T]$ is an interval on which $\psi(t) \geq M$, then ψ is bounded below pointwise by the solution to the initial value problem

$$\left.\begin{aligned} \ddot{w}(t) &= (\alpha/2)w(t) \\ w(0) &= \psi(0) \\ \dot{w}(0) &= \dot{\psi}(0). \end{aligned}\right\} \tag{2.5.110}$$

Putting $a = \psi(0) = \left|v(0)\right|^2$, $b = \dot{\psi}(0) = \big\langle v(0),\, \dot{v}(0)\big\rangle = \big\langle v(0),\, \xi\big\rangle \leq 0$, and $k = \sqrt{\alpha/2}$, the solution to this initial value problem is

$$w(t) = \frac{1}{2}\left(a + \frac{b}{k}\right)e^{kt} + \frac{1}{2}\left(a - \frac{b}{k}\right)e^{-kt}. \tag{2.5.111}$$

It is clear that if a is sufficiently large depending on b and k, then $w(t) \geq M$ for all $t \geq 0$. This implies that if $\left|v(0)\right| > M$ is sufficiently large depending on $\left|\xi\right|$ and α, then $\psi(t) > w(t) > M$ on every interval $[0,\, T]$. Since $w(t) \to \infty$ as $t \to \infty$ when a is large, this implies that for sufficiently large $\left|v(0)\right|$ (relative to $\dot{v}(0) = \xi$), we have $\psi(t) \to \infty$ as $t \to \infty$. This being impossible when $v \in C_{\{0\}}$, we have proved the existence of a bound for $\left|v(0)\right|$, depending only on ξ (and G, via α). This completes the verification of the hypotheses of Theorem 2.4.1 on page 83 in the case of the Neumann problem, and we state the result here.

Example 2.5.12. Suppose that G satisfies all of the stated conditions, starting on page 102 and ending just before (2.5.69). Then for each $g \in C_{\{0\}}\big([0,\,\infty),\,\mathbb{R}^d\big)$ and $\xi \in \mathbb{R}^d$, the Neumann problem

$$\left.\begin{aligned} \ddot{v}(t) + G\big(t,\,v(t)\big) = g(t) \qquad &\forall t \geq 0; \\ \dot{v}(0) = \xi \qquad\qquad & \end{aligned}\right\} \tag{2.5.112}$$

has a solution $v \in C^2\big([0,\,\infty),\,\mathbb{R}^d\big)$ such that $\left|v(t)\right|$, $\left|\dot{v}(t)\right|$, and $\left|\ddot{v}(t)\right|$ all tend to 0 as $t \to \infty$.

♦

3.0 PARTIAL DIFFERENTIAL EQUATIONS IN UNBOUNDED CYLINDERS

Let Ω be a bounded domain in $\mathbb{R}^d$, and let I be an interval, bounded or not. Take a real number $p \in (d+1, \infty)$. As a result, $W_0^{1,p}(I \times \Omega)$ embeds in the Hölder space $C^{0,\lambda}(\overline{I \times \Omega})$ for some $\lambda \in (0, 1)$, and functions in $W_0^{1,p}(I \times \Omega)$ are defined pointwise on $\overline{I \times \Omega}$. See the Sobolev Embedding Theorem in Adams and Fournier [AF03], especially Parts II and III of Theorem 4.12.

We are concerned with boundary value problems of the following type:

$$u_t(t, x) - A(t)u(t, x) - G\big(t, u(t, x)\big) = f(t, x), \qquad t \geq 0,\ x \in \Omega; \tag{3.0.1a}$$

$$u(0, x) = g(x), \qquad x \in \Omega; \tag{3.0.1b}$$

$$u(t, x) = 0, \qquad t \geq 0,\ x \in \partial\Omega, \tag{3.0.1c}$$

$$\lim_{t\to\infty} \sup_{x\in\Omega} \big|u(t, x)\big| = 0. \tag{3.0.1d}$$

Here, f and g are given functions (drawn from a space to be determined), $A(t)$ is a differential operator in Ω, and $G = G(t, \xi)$ is some nonlinearity.

Remark 3.0.13. We have chosen to keep G independent of $x \in \Omega$ only for convenience. Allowing $G = G(t, x, \xi)$ would seem to present little or no difficulty, as long as the key hypotheses about G are rephrased in an appropriate way. For example, instead of assuming that $G = G(t,\, \xi)$ is uniformly continuous on $[0,\, \infty) \times K$ for each compact K, one would assume that $G = G(t,\, x,\, \xi)$ is uniformly continuous on $[0,\, \infty) \times \Omega \times K$. $\Diamond$

Our first task will be to identify appropriate function spaces such that problem (3.0.1) can be re-interpreted as an evolution equation involving vector valued functions on the half-line. We then adapt the program of Chapter 2 to prove the existence of solutions to problem (3.0.1). Of course, there will be some difficulties not present in Chapter 2 because the ambient spaces are infinite dimensional. On the other hand, there is some additional structure afforded by the fact that these spaces are function spaces rather than more general Banach spaces.

3.1 TWO EQUIVALENT FUNCTIONAL SETTINGS

Remark 3.1.1. The results of Section 3.1 are standard, and may be skimmed over by the reader who is comfortable with them. In particular, this applies to the reader who is comfortable with the identification of $W^{1,\,p}\big(I,\, L^p(\Omega)\big) \cap L^p\big(I,\, W^{2,\,p}(\Omega)\big)$ with that subspace of $W^{1,\,p}(I \times \Omega)$ consisting of functions whose second-order spacial derivatives are in $L^p(I \times \Omega)$. The material is included for completeness, as such identifications are central to our approach.

$\Diamond$

We adopt the following notation:

$$X^p := L^p(\Omega) \tag{3.1.1}$$

$$W^p := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \tag{3.1.2}$$

$$\mathscr{X}^p(I) := L^p(I,\, X^p) \tag{3.1.3}$$

$$\mathscr{W}^p(I) := W^{1,p}(I,\, X^p) \cap L^p(I,\, W^p) \tag{3.1.4}$$

$$\mathscr{W}_0^p(I) = W_0^{1,p}(I,\, X^p) \cap L^p(I,\, W^p) \tag{3.1.5}$$

We also will make use of the following spaces of (equivalence classes of) functions on the cylinder $I \times \Omega$:

$$\widehat{\mathscr{X}^p}(I) := L^p(I \times \Omega) \tag{3.1.6}$$

$$\begin{aligned}\widehat{\mathscr{W}^p}(I) := \big\{ u \in W^{1,p}\big(I \times \Omega\big) \,:\, & D^\alpha u \in L^p(I \times \Omega),\, \forall \left|\alpha\right| \le 2; \\ & u(t,\, x) = 0 \text{ for all } x \in \partial\Omega \text{ and all } t \ge 0 \big\}\end{aligned} \tag{3.1.7}$$

Here, $\alpha \in \mathbb{Z}^{d+1}$ is the usual multi-index notation, except that D^α is reserved for derivatives in the x variables. Notice that $W_0^{2,p}(I \times \Omega) \subset \widehat{\mathscr{W}^p}(I) \subset W^{1,p}(I \times \Omega)$, with strict inclusions. We equip $\widehat{\mathscr{W}^p}(I)$ with the norm

$$\left\|u\right\|_{\widehat{\mathscr{W}^p}(I)} := \left(\left\|u\right\|^p_{L^p(I\times\Omega)} + \left\|\frac{\partial u}{\partial t}\right\|^p_{L^p(I\times\Omega)} + \sum_{\alpha \le 2} \left\|D^\alpha u\right\|^p_{L^p(I\times\Omega)} \right)^{1/p}. \tag{3.1.8}$$

In the literature, a notation such as $W^{(1,2,\ldots,2),(p,p,\ldots p)}(I \times \Omega)$ is sometimes used to identify clearly the conditions associated with each variable. The fact that the conditions are not the same for each variable is why the space $\widehat{\mathscr{W}^p}(I)$ is known as an anisotropic Sobolev space. For a general treatment of such spaces see Besov, Il′in and Nikol′skiĭ [BIN78, BIN79], especially the third chapter in the first volume. Before turning to the relationships between the various spaces introduced here, we prove the following embedding result for $\widehat{\mathscr{W}^p}(I)$, which we will use often:

Lemma 3.1.2. *The space $\widehat{\mathscr{W}^p}(I)$ embeds in $C^{0,\lambda}(\overline{I \times \Omega})$ for some $0 < \lambda < 1$.*

Proof. Begin by extending I to $I' = I + [-1, 1]$. We have the embedding $W_0^{1,p}(I' \times \Omega)$ in $C^{0,\lambda}(\overline{I' \times \Omega})$, as mentioned above. It remains only to show that there is a continuous extension operator from $\widehat{\mathscr{W}^p}(I)$ into $W_0^{1,p}(I' \times \Omega)$. If $I = I' = \mathbb{R}$, there is nothing to do, since $W^{1,p}(\mathbb{R}) = W_0^{1,p}(\mathbb{R})$. All other cases may be illustrated by the treatment of the case $I = (0, \infty)$ and $I' = (-1, \infty)$. For each $u \in \widehat{\mathscr{W}^p}(I)$, we define $v : I' \times \Omega \to \mathbb{R}$ as follows:

$$v(t, x) := \begin{cases} u(t, x) & \text{if } t \geq 0 \\ (t+1)u(-t, x) & \text{if } -1 \leq t < 0. \end{cases} \tag{3.1.9}$$

It is clear that $v \in W_0^{1,p}(I' \times \Omega)$, with

$$\frac{\partial v}{\partial x_i}(t, x) := \begin{cases} \frac{\partial u}{\partial x_i}(t, x) & \text{if } t \geq 0 \\ (t+1)\frac{\partial u}{\partial x_i}(-t, x) & \text{if } -1 \leq t < 0 \end{cases} \tag{3.1.10}$$

and

$$\frac{\partial v}{\partial t}(t, x) := \begin{cases} \frac{\partial u}{\partial t}(t, x) & \text{if } t \geq 0 \\ u(-t, x) - (t+1)\frac{\partial u}{\partial t}(-t, x) & \text{if } -1 \leq t < 0. \end{cases} \tag{3.1.11}$$

It follows that

$$\|v\|_{W^{1,p}(I' \times \Omega)} \leq C \|u\|_{W^{1,p}(I \times \Omega)} \leq C \|u\|_{\widehat{\mathscr{W}^p}(I)}. \tag{3.1.12}$$

$\square$

Remark 3.1.3. Our choice of notation in equations (3.1.6) and (3.1.7) suggests a strong relationship between $\mathscr{X}^p(I)$ and $\widehat{\mathscr{X}^p}(I)$ and between $\mathscr{W}^p(I)$ and $\widehat{\mathscr{W}^p}(I)$; we shall indeed see that there is a canonical (if slightly technical) isometry between each pair. In order to be justified in using the spaces interchangeably in the sequel, we shall also verify that all of the relevant operations are compatible under the isometry. For example, we want not only to know that $\mathbf{u} = \mathbf{u}(t) \in \mathscr{W}^p(I)$ is in correspondence with some $u = u(t, x) \in \widehat{\mathscr{W}^p}(I)$, but also that $\frac{\mathrm{d}}{\mathrm{d}t} : \mathscr{W}^p(I) \to \mathscr{X}^p(I)$ stands in the desired relationship with $\frac{\partial}{\partial t} : \widehat{\mathscr{W}^p}(I) \to \widehat{\mathscr{X}^p}(I)$.

However, we *cannot* begin with the declaration that each $\mathbf{u} \in \widehat{\mathscr{X}^p}(I) = L^p(I, L^p(\Omega))$ corresponds canonically to a function $u : I \times \Omega \to \mathbb{R}$ via the "definition" $u(t, x) = \mathbf{u}(t)(x)$. Firstly, each element of $\mathscr{X}^p(I)$ is not a function $\mathbf{u}$ defined on I, but rather an equivalence class $[\mathbf{u}]$ of such functions. Secondly, the values $\mathbf{u}(t)$ of these functions are are in $L^p(\Omega)$, and so are not themselves functions f defined on Ω, but again equivalence classes $[f]$ of such functions. Such matters are often trivialities that are bypassed by showing that it is enough to choose any representative for each class encountered. However, if we select a representative function $\mathbf{u}$ for $[\mathbf{u}] \in \mathscr{X}^p(I)$, and then further select representative functions $f(t) \in [\mathbf{u}(t)]$ on Ω for each $t \in I$, and then take $u(t, x)$ to be $f(t)(x)$, the so obtained function u on $I \times \Omega$ may not even be Lebesgue measurable! This can actually happen if one proceeds in such a manner, as is illustrated by the following example (found in a slightly different context in Rudin[Rud87] and due to Sierpinski).

Example 3.1.4. Take $I = (0, 1)$. Assuming that the continuum hypothesis holds, there is a well-ordering $\preceq$ of I such that each $t \in I$ has at most countably many $\preceq$-predecessors. Take also $\Omega = (0, 1) \subset \mathbb{R}$. We define a function u on $I \times \Omega$ by declaring that $u(t, x) = 0$ if $t \preceq x$, and $u(t, x) = 1$ otherwise. Then, for each $t \in I$, $u(t, x) = 0$ for almost every $x \in I$.

Hence, for each $t \in I$, $u(t, \cdot) = 0 \in L^p(\Omega)$, and hence u is a "selection", in the sense of the previous paragraph, for $\mathbf{0} \in \mathscr{X}^p(I)$. However, u is not Lebesgue measurable on $I \times \Omega$. This can be seen by appealing to Fubini's theorem, as follows. The above discussion shows that $\int_I \int_\Omega u(t, x)\, \mathrm{d}x\, \mathrm{d}t = 0$. By similar considerations, for each $x \in \Omega$, $u(t, x) = 1$ for almost every $t \in I$, implying that $\int_\Omega \int_I u(t, x)\, \mathrm{d}t\, \mathrm{d}x = 1$. By Fubini's Theorem, this is impossible if u is measurable. ♦

In light of this example, we must proceed carefully, first by carefully defining the selection process used above, and then by using it carefully. Until this study is complete, we will explicitly display elements of L^p spaces as equivalence classes. We shall always use the notation $[u]$ to stand for the equivalence class of functions equal to u "almost everywhere", but of course it must be understood from context whether "almost everywhere" means for almost every $t \in I$, almost every $x \in \Omega$, or almost every $(t, x) \in I \times \Omega$. ◊

Definition 3.1.5. Let $w : I \times \Omega \to \mathbb{R}$ and $[\mathbf{u}] \in \mathscr{X}^p(I)$. For each $t \in I$, define the function $f_t : \Omega \to \mathbb{R}$ by $f_t(x) := w(t, x)$. Hence, for each $t \in I$, f_t is a function on Ω, while $\mathbf{u}(t) \in L^p(\Omega)$ is an equivalence class of such functions. If $f_t \in \mathbf{u}(t)$ for almost every $t \in I$, then w is said to be a selection for $[\mathbf{u}]$.

It is clear enough that selection is well-defined, in the sense that it does not depend on the choice of representative for the equivalence class $[\mathbf{u}]$. Before we turn to questions of existence, we prove some properties of those selections that are Lebesgue measurable on $I \times \Omega$.

Lemma 3.1.6 (Linearity). *If w_i is a measurable selection for $\mathbf{u}_i$, $i = 1, 2$, and if $s \in \mathbb{R}$, then $w_1 + sw_2$ is a measurable selection for $[\mathbf{u}_1] + s[\mathbf{u}_2]$.*

Proof. Define, for $i = 1, 2$ and each $t \in I$, $(f_i)_t : \Omega \to \mathbb{R}$ by $(f_i)_t(x) = w_i(t, x)$. Then $(f_i)_t \in \mathbf{u}_i(t)$ for both $i = 1$ and $i = 2$, except for those t belonging to a set of measure zero. Using the vector operations on the equivalence classes that make up a quotient space, we have then that $(f_1)_t + s(f_2)_t \in \mathbf{u}_1(t) + s\mathbf{u}_2(t)$, for almost every t. This proves that $w_1 + sw_2$ is a selection for $[\mathbf{u}_1 + s\mathbf{u}_2] = [\mathbf{u}_1] + s[\mathbf{u}_2]$. The measurability of $w_1 + sw_2$ follows immediately from the measurability of w_1 and of w_2. □

Lemma 3.1.7 (Isometry). *If w is a measurable selection for $[\mathbf{u}] \in \mathscr{X}^p(I) = L^p\big(I,\, L^p(\Omega)\big)$, then $[w] \in L^p(I \times \Omega)$ with $\big\|[w]\big\|_{L^p(I\times\Omega)} = \big\|[\mathbf{u}]\big\|_{\mathscr{X}^p(I)}$.*

Proof. For each $t \in I$, define $f_t : \Omega \to \mathbb{R}$ by $f_t(x) := w(t, x)$. By the Fubini Theorem,

$$\int_{I\times\Omega} \big|w(t,\, x)\big|^p \,\mathrm{d}(t,\, x) = \int_I \left(\int_\Omega \big|f_t(x)\big|^p \,\mathrm{d}x \right) \mathrm{d}t \tag{3.1.13}$$

$$= \int_I \big\|\mathbf{u}(t)\big\|^p_{L^p(\Omega)} \,\mathrm{d}t \tag{3.1.14}$$

$$= \big\|[\mathbf{u}]\big\|^p_{\mathscr{X}^p(I)}\,, \tag{3.1.15}$$

which proves both assertions. □

We now recall the following result, which is Theorem III.11.17 of Dunford and Schwartz [DS88], specialized to the current situation.

Lemma 3.1.8 ($\mathbf{L^1}$ Existence). *Let $[\mathbf{v}] \in L^1\big(I,\, L^1(\Omega)\big)$. Then there exists a measurable selection $w : I \times \Omega \to \mathbb{R}$ for $[\mathbf{v}]$, and this selection is unique up to subsets of $I \times \Omega$ of Lebesgue measure zero. Moreover, $w(\cdot,\, x)$ is integrable on I for almost all $x \in \Omega$, and if a map $h : \Omega \to \mathbb{R}$ is such that*

$$h(x) = \int_I w(t,\, x)\; dt \tag{3.1.16}$$

whenever the integral exists, then

$$h \in \int_I \mathbf{v}(t)\, dt \in L^1\big(I,\, L^1(\Omega)\big). \quad \textit{(Bochner integral)} \tag{3.1.17}$$

Lemma 3.1.9 (Existence). *Let* $[\mathbf{u}] \in \mathscr{X}^p(I)$*. Then there exists a measurable selection* $w : I \times \Omega \to \mathbb{R}$ *for* $[\mathbf{u}]$*.*

Proof. To apply Lemma 3.1.8, we decompose $\mathbf{u}$ into nonnegative and non-positive parts $\mathbf{u} = \mathbf{u}_+ - \mathbf{u}_-$. This is done as usual:

$$\mathbf{u}_\pm = \frac{1}{2}\left(\left|\mathbf{u}\right| \pm \mathbf{u}\right). \tag{3.1.18}$$

(The absolute value function is applied pointwise.) By pointwise composition with the p^{th}-power function, we obtain functions $\mathbf{v}_+ = \mathbf{u}_+^p$ and $\mathbf{v}_- = \mathbf{u}_-^p$; these are members of $L^1\big(I,\, L^1(\Omega)\big)$. Hence, according to Lemma 3.1.8, there are measurable selections w_+ and w_- for $\mathbf{v}_+$ and $\mathbf{v}_-$, respectively. By possibly redefining w_+ and w_- on a set of measure zero, these functions are nonnegative. Hence, $w = w_+^{1/p} - w_-^{1/p}$ is well-defined.

We claim that w is a measurable selection for $[\mathbf{u}]$. The measurability follows from the measurability of w_+ and w_-. Finally, almost every $t \in I$ is such that $w_+(t,\, \cdot) \in \mathbf{v}_+$ and $w_-(t,\, \cdot) \in \mathbf{v}_-(t)$. For such t,

$$w(t,\, \cdot) = \big(w_+(t,\, \cdot)\big)^{1/p} - \big(w_-(t,\, \cdot)\big)^{1/p} \tag{3.1.19}$$

$$\in \mathbf{v}_+^{1/p} - \mathbf{v}_-^{1/p} \tag{3.1.20}$$

$$= \mathbf{u}_+(t) - \mathbf{u}_-(t) = \mathbf{u}(t). \tag{3.1.21}$$

□

And now we turn to uniqueness:

Lemma 3.1.10 (Uniqueness). *Let w_1 and w_2 be measurable selections of $[\mathbf{u}] \in \mathscr{X}^p(I)$. Then $[w_1] = [w_2]$.*

Proof. According to Lemma 3.1.6, $w_1 - w_2$ is a measurable selection of $\mathbf{0} \in \mathscr{X}^p(I)$. Hence, according to Lemma 3.1.7, $[w_1 - w_2]$ has norm zero in $L^p(I \times \Omega)$. Hence, $0 = [w_1 - w_2] = [w_1] - [w_2]$, as desired. □

The following definition is now justified:

Definition 3.1.11 (The Selector "J"). Let $[\mathbf{u}] \in \mathscr{X}^p(I)$. Let $w : I \times \Omega \to \mathbb{R}$ be any measurable selection for $[\mathbf{u}]$. We say that $[w]$ represents $[\mathbf{u}]$ in $\widehat{\mathscr{X}^p}(I) = L^p(I \times \Omega)$, and we write $J([\mathbf{u}]) := [w]$. (The symbol "J" will be reserved for this use in the sequel.)

Theorem 3.1.12 (Isometry). *The map J is a linear isometry of the Banach space $\mathscr{X}^p(I) = L^p\big(I,\, L^p(\Omega)\big)$ onto the Banach space $\widehat{\mathscr{X}^p}(I) = L^p(I \times \Omega)$.*

Proof. Lemmas 3.1.6 and 3.1.7 together imply that J is a linear isometry onto some subspace of $\widehat{\mathscr{X}^p}(I)$. It remains only to show that J is onto $\widehat{\mathscr{X}^p}(I)$. Given $[w] \in L^p(I \times \Omega)$, it follows from Fubini's Theorem that almost all of the functions $w(t,\, \cdot)$, $t \in I$, are in $L^p(\Omega)$. Hence, a function $\mathbf{u} : I \to L^p(\Omega)$ is well-defined almost everywhere by setting $\mathbf{u}(t) = [w(t,\, \cdot)]$. Moreover, by part (b) of Lemma III.11.16 of Dunford and Schwartz[DS88], the function $\mathbf{u}$ is measurable.

To see that $[\mathbf{u}] \in \mathscr{X}^p(I) = L^p\big(I,\, L^p(\Omega)\big)$, we calculate

$$\int_I \big\|\mathbf{u}(t)\big\|^p_{L^p(\Omega)}\,\mathrm{d}t = \int_I \big\|[w(t,\, \cdot)]\big\|^p_{L^p(\Omega)}\,\mathrm{d}t \tag{3.1.22}$$

$$= \int_I \left(\int_\Omega \big|w(t,\, x)\big|^p\,\mathrm{d}x\right)\mathrm{d}t \tag{3.1.23}$$

$$= \int_{I\times\Omega} \big|w(t,\, x)\big|^p\,\mathrm{d}(t,\, x) < \infty, \tag{3.1.24}$$

with use of Fubini's Theorem. Finally, it is immediate from the construction of $\mathbf{u}$ that w is a measurable selection of $[\mathbf{u}]$. Hence, $[w] = J([\mathbf{u}])$, which completes this proof. $\square$

We now investigate the compatibility of differentiation of $[\mathbf{u}] \in L^p\big(I,\, L^p(\Omega)\big)$ and of $u = J([\mathbf{u}]) \in L^p(I \times \Omega)$. There are two sorts of differentiability at issue here. On one hand, $[\mathbf{u}]$ is a locally integrable Banach space valued function of the single variable $t \in I$, and may be differentiated, in the sense of distributions, with respect to t. On the other hand, each representative function $\mathbf{v} \in [\mathbf{u}]$ has values in the space $L^p(\Omega)$, and these values have distributional derivatives with respect to the variables x_i. We shall see that both correspond in a canonical, isometric way, with the partial derivatives of $u = J[\mathbf{u}]$.

We first consider the derivative with respect to the t variable. The result is:

Lemma 3.1.13 ($\frac{\partial}{\partial \mathbf{t}} \circ J$). *Suppose that $[\mathbf{u}] \in W^{1,p}\big(I,\, L^p(\Omega)\big)$, so that the distributional derivative $[\mathbf{v}] = \frac{d}{dt}[\mathbf{u}]$ is a well-defined member of $\mathscr{X}^p(I)$. Let $[u] = J([\mathbf{u}])$ and $[v] = J([\mathbf{v}])$. Then $\frac{\partial}{\partial t}[u] = [v]$. In other words, $J\,\frac{d}{\mathrm{d}t} = \frac{\partial}{\partial t} J$ as operators on $W^{1,p}\big(I,\, L^p(\Omega)\big)$.*

Conversely, if $[u] \in L^p(I \times \Omega)$ is such that the distributional derivative $\frac{\partial}{\partial t}[u]$ is also in $L^p(I \times \Omega)$, then $J^{-1}[u] \in W^{1,p}\big(I,\, L^p(\Omega)\big)$, with $\frac{d}{\mathrm{d}t}\, J^{-1}[u] = J^{-1}\frac{\partial}{\partial t}[u]$.

Proof. To show that $\frac{\partial}{\partial t}[u] = [v]$ in the sense of distributions, it suffices to check that for all $\phi \in C_0^\infty\big(I\big)$ and $\psi \in C_0^\infty\big(\Omega\big)$, that

$$\int_{I\times\Omega} u(t,\, x)\phi'(t)\psi(x)\,\mathrm{d}(t,\, x) = -\int_{I\times\Omega} v(t,\, x)\phi(t)\psi(x)\,\mathrm{d}(t,\, x). \tag{3.1.25}$$

To see that it is valid to consider only the test functions of the form $\phi(t)\psi(x)$, see Theorem 5.2.1 (the Schwartz kernel theorem) in Hörmander[Hör90]. We first apply the Fubini theorem:

$$\int_{I\times\Omega} u(t,\, x)\phi'(t)\psi(x)\,\mathrm{d}(t,\, x) = \int_\Omega \left(\int_I u(t,\, x)\phi'(t)\,\mathrm{d}t\right)\psi(x)\,\mathrm{d}x. \tag{3.1.26}$$

Next, we apply the final conclusion of Lemma 3.1.8 to the inner integral, thus replacing it by a Bochner integral:

$$\int_\Omega \left(\int_I u(t,\, x)\phi'(t)\,\mathrm{d}t \right) \psi(x)\,\mathrm{d}x = \int_\Omega \left(\int_I \mathbf{u}(t)\phi'(t)\,\mathrm{d}t \right)(x)\psi(x)\,\mathrm{d}x \tag{3.1.27}$$

By applying similar considerations to the integral $-\int_{I\times\Omega} v(t,\, x)\phi(t)\psi(x)\,\mathrm{d}(t,\, x)$, and by using the fact that $[\mathbf{v}] = \frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{u}]$ in the first place, we conclude that

$$\begin{aligned} \int_{I\times\Omega} u(t,\, x)\phi'(t)\psi(x)\,\mathrm{d}(t,\, x) &= \int_\Omega \left(\int_I \mathbf{u}(t)\phi'(t)\,\mathrm{d}t \right)(x)\psi(x)\,\mathrm{d}x & (3.1.28)\\ &= -\int_\Omega \left(\int_I \mathbf{v}(t)\phi(t)\,\mathrm{d}t \right)(x)\psi(x)\,\mathrm{d}x & (3.1.29)\\ &= -\int_{I\times\Omega} v(t,\, x)\phi(t)\psi(x)\,\mathrm{d}(t,\, x), & (3.1.30) \end{aligned}$$

as desired. For the converse, we take $[\mathbf{u}] := J^{-1}[u]$ and $[\mathbf{v}] := J^{-1}\frac{\partial}{\partial t}[u]$. To show that $[\mathbf{u}] \in W^{1,p}\big(I,\, L^p(\Omega)\big)$, we show that for all $\phi \in C_0^\infty\big(I\big)$,

$$\int_I \mathbf{u}(t)\phi'(t)\,\mathrm{d}t = -\int_I \mathbf{v}(t)\phi(t)\,\mathrm{d}t \quad \text{(Bochner integrals)} \tag{3.1.31}$$

as elements of $L^p(\Omega)$. It suffices to check that both sides of (3.1.31) have the same action on test functions $\psi \in C_0^\infty\big(\Omega\big)$, which is to verify that for all $\psi \in C_0^\infty\big(\Omega\big)$,

$$\int_\Omega \left(\int_I \mathbf{u}(t)\phi'(t)\,\mathrm{d}t \right)(x)\psi(x)\,\mathrm{d}x = -\int_\Omega \left(\int_I \mathbf{v}(t)\phi(t)\,\mathrm{d}t \right)(x)\psi(x)\,\mathrm{d}x. \tag{3.1.32}$$

This is verified by using the Fubini theorem and Lemma 3.1.8, just as in the first half of the proof:

$$\int_\Omega \left(\int_I \mathbf{u}(t)\phi'(t)\,\mathrm{d}t\right)(x)\psi(x)\,\mathrm{d}x = \int_\Omega \left(\int_I u(t,\,x)\phi'(t)\,\mathrm{d}t\right)\psi(x)\,\mathrm{d}x \tag{3.1.33}$$

$$= \int_{I\times\Omega} u(t,\,x)\phi'(t)\psi(x)\,\mathrm{d}(t,\,x) \tag{3.1.34}$$

$$= -\int_{I\times\Omega} \frac{\partial}{\partial t}u(t,\,x)\phi(t)\psi(x)\,\mathrm{d}(t,\,x) \tag{3.1.35}$$

$$= -\int_\Omega \left(\int_I \frac{\partial}{\partial t}u(t,\,x)\phi'(t)\,\mathrm{d}t\right)\psi(x)\,\mathrm{d}x \tag{3.1.36}$$

$$= -\int_\Omega \left(\int_I \mathbf{v}(t)\phi'(t)\,\mathrm{d}t\right)(x)\psi(x)\,\mathrm{d}x. \tag{3.1.37}$$

□

We now turn to differentiation with respect to the coordinates $x_1,\, x_2,\, \ldots,\, x_d$ of $x \in \Omega$. Begin with the following observation. Suppose that $[\mathbf{u}] \in L^p\big(I,\, W^{2,p}(\Omega)\big)$. Since we are regarding $L^p\big(I,\, W^{2,p}(\Omega)\big)$ as a subspace of $L^p\big(I,\, L^p(\Omega)\big)$, this implies that $\mathbf{u}(t) \in W^{2,p}(\Omega)$ for almost every $t \in I$. Since D^α is a bounded linear operator from $W^{2,p}(\Omega)$ to $L^p(\Omega)$ for each $\big|\alpha\big| \le 2$, the pointwise application of D^α to $\mathbf{u} = \mathbf{u}(t)$ results in an almost everywhere well defined measurable function $\mathbf{v}_\alpha : I \to L^p(\Omega)$ given by $\mathbf{v}_\alpha(t) := D^\alpha(\mathbf{u}(t))$, such that $[\mathbf{v}_\alpha] \in \mathscr{X}^p(I)$. Notice that although $\mathbf{v}_\alpha$ depends on the choice of $\mathbf{u} \in [\mathbf{u}]$, the equivalence class $[\mathbf{v}_\alpha]$ is independent of this choice. We denote the map $[\mathbf{u}] \mapsto [\mathbf{v}_\alpha]$ by $\partial_\alpha : L^p\big(I,\, W^{2,p}(\Omega)\big) \to \mathscr{X}^p(I)$.

Lemma 3.1.14 ($\mathbf{D}^\alpha \circ \mathbf{J}$). *Suppose that $[\mathbf{u}] \in L^p\big(I,\, W^{2,p}(\Omega)\big)$, and that $\big|\alpha\big| \le 2$. Then $D^\alpha J[\mathbf{u}] = J\partial_\alpha[\mathbf{u}] \in L^p(I \times \Omega)$, in the sense of distributions in $I \times \Omega$. In other words, $D^\alpha J = J\partial_\alpha$ as maps from $L^p\big(I,\, W^{2,p}(\Omega)\big)$ into $L^p(I \times \Omega)$.*

Conversely, if $[u] \in L^p(I \times \Omega)$ is such that for each $\big|\alpha\big| \le 2$, the distributional derivative $D^\alpha[u]$ is also in $L^p(I \times \Omega)$, then $J^{-1}[u] \in L^p\big(I,\, W^{2,p}(\Omega)\big)$, with $\partial_\alpha J^{-1}[u] = J^{-1}D^\alpha[u]$.

Proof. For the forward implication, we use $[u] = J[\mathbf{u}]$ and $[v] = J\partial_\alpha[\mathbf{u}]$. It follows from the definitions for J and for ∂_α that for almost every $t \in I$, $\mathbf{u}(t) \in W^{2,p}(\Omega)$, $u(t,\,\cdot) \in \mathbf{u}(t)$ and $v(t,\,\cdot) \in D^\alpha(\mathbf{u}(t))$. This implies that for almost every $t \in I$ and for every $\psi \in C_0^\infty\big(\Omega\big)$, one has that

$$\int_\Omega u(t,\,x) D^\alpha \psi(x)\,\mathrm{d}x = (-1)^{\,\alpha}\int_\Omega v(t,\,x)\psi(x)\,\mathrm{d}x. \tag{3.1.38}$$

Using Fubini's Theorem (recall that u and v are Lebesgue measurable), this implies that for almost every $t \in I$, for all $\psi \in C_0^\infty\big(\Omega\big)$ and for all $\phi \in C_0^\infty\big(I\big)$,

$$\int_{I\times\Omega} u(t,\,x)\phi(t) D^\alpha \psi(x)\,\mathrm{d}(t,\,x) = (-1)^{\,\alpha}\int_{I\times\Omega} v(t,\,x)\phi(t)\psi(x)\,\mathrm{d}(t,\,x). \tag{3.1.39}$$

Hence, $[v] = D^\alpha[u] = D^\alpha J[\mathbf{u}]$, as claimed. As in the proof of Lemma 3.1.13, see Theorem 5.2.1 (the Schwartz kernel theorem) in Hörmander[Hör90] to see that it is valid to consider only the test functions of the form $\phi(t)\psi(x)$,

For the converse, suppose that $D^\alpha[u] \in L^p(I\times\Omega)$ for each $\big|\alpha\big| \le 2$, put $[\mathbf{u}] = J^{-1}[u]$, and let $v_\alpha \in D^\alpha[u] \in L^p(I\times\Omega)$. Let $\phi \in C_0^\infty\big(I\big)$ and $\psi \in C_0^\infty\big(\Omega\big)$ be given. First, we have that

$$\int_{I\times\Omega} u(t,\,x)\phi(t) D^\alpha \psi(x)\,\mathrm{d}(t,\,x) = (-1)^{\,\alpha}\int_{I\times\Omega} v_\alpha(t,\,x)\phi(t)\psi(x)\,\mathrm{d}(t,\,x). \tag{3.1.40}$$

From Fubini's theorem, it then follows that

$$\int_I \left(\int_\Omega u(t,\,x) D^\alpha \psi(x)\,\mathrm{d}x\right)\phi(t)\,\mathrm{d}t = (-1)^{\,\alpha}\int_I \left(\int_\Omega v_\alpha(t,\,x)\psi(x)\,\mathrm{d}x\right)\phi(t)\,\mathrm{d}t. \tag{3.1.41}$$

Since $\phi \in C_0^\infty\big(I\big)$ was arbitrary, this means that we have the following equality, in the sense of distributions on I:

$$\int_\Omega u(\cdot,\,x) D^\alpha \psi(x)\,\mathrm{d}x = (-1)^{\,\alpha}\int_\Omega v_\alpha(\cdot,\,x)\psi(x)\,\mathrm{d}x. \tag{3.1.42}$$

Since equality of functions in the sense of distributions implies pointwise equality almost everywhere, we have

$$\int_\Omega u(t,\, x) D^\alpha \psi(x)\, \mathrm{d}x = (-1)^{\ \alpha} \int_\Omega v_\alpha(t,\, x)\psi(x)\, \mathrm{d}x, \tag{3.1.43}$$

for almost all $t \in I$. Since $\psi \in C_0^\infty\big(\Omega\big)$ is arbitrary, this proves that

$$D^\alpha[u(t,\, \cdot)] = [v_\alpha(t,\, \cdot)], \tag{3.1.44}$$

for almost all $t \in I$, in the sense of distributions on Ω. Since $[v_\alpha(t,\, \cdot)] \in L^p(\Omega)$ for almost all $t \in I$ (by Fubini's Theorem) and since $[u(t,\, \cdot)] = \mathbf{u}(t)$, we have shown that $\mathbf{u}(t) \in W^{2,\,p}(\Omega)$ for almost all $t \in I$. It remains only to verify that

$$\int_I \big\|\mathbf{u}(t)\big\|^p_{W^{2,\,p}(\Omega)}\, \mathrm{d}t < \infty. \tag{3.1.45}$$

Indeed, for each $\big|\alpha\big| \leq 2$,

$$\int_I \big\|D^\alpha(\mathbf{u}(t))\big\|^p_{L^p(\Omega)}\, \mathrm{d}t = \int_I \big\|D^\alpha[u(t,\, \cdot)]\big\|^p_{L^p(\Omega)}\, \mathrm{d}t \tag{3.1.46}$$

$$= \int_I \int_\Omega \big|v_\alpha(t,\, x)\big|^p\, \mathrm{d}x\, \mathrm{d}t \tag{3.1.47}$$

$$= \big\|D^\alpha u\big\|^p_{L^p(I\times\Omega)} < \infty. \tag{3.1.48}$$

$\square$

Theorem 3.1.15 (Isometry and Commutativity). *The restriction of the measurable selection map J to $\mathscr{W}^p(I)$ is an isometric isomorphism of $\mathscr{W}^p(I)$ onto $\widehat{\mathscr{W}}^p(I)$, such that the following diagrams commute, the second for each $\big|\alpha\big| \leq 2$:*

$$\begin{tikzcd} \mathscr{W}^p(I) \arrow[r, "\frac{d}{dt}"] \arrow[d, "J"] & \mathscr{X}^p(I) \arrow[d, "J"] \\ \widehat{\mathscr{W}}^p(I) \arrow[r, "\frac{\partial}{\partial t}"] & \widehat{\mathscr{X}}^p(I) \end{tikzcd} \qquad \begin{tikzcd} \mathscr{W}^p(I) \arrow[r, "\partial_\alpha"] \arrow[d, "J"] & \mathscr{X}^p(I) \arrow[d, "J"] \\ \widehat{\mathscr{W}}^p(I) \arrow[r, "D^\alpha"] & \widehat{\mathscr{X}}^p(I) \end{tikzcd}$$

Proof. This is a direct consequence of Lemmas 3.1.13 and 3.1.14. $\square$

Remark 3.1.16 *(Notation).* Having established clearly the relationship between the objects in the various spaces under consideration, we will resume the usual suppression of the equivalence class notation $[\cdot]$. $\Diamond$

We pause to prove a useful result on the decay of functions in $\widehat{\mathscr{W}^p}(I)$ as $|t| \to \infty$. The proof also serves as an illustration of the utility of the isometry J; some properties are readily available for objects in $\mathscr{W}^p(I)$, while others are more readily available to object in $\widehat{\mathscr{W}^p}(I)$. Hence the benefit of knowing that these spaces are the same.

Lemma 3.1.17 (Decay at Infinity). *Let $R > 0$. There are constants $C = C(R) > 0$ and $\mu \in (0, 1)$ such that for each $\mathbf{u} \in \mathscr{W}^p\big([0, \infty)\big)$, if $\|u\|_{\mathscr{W}^p\left([0,\infty)\right)} \leq R$, then*

$$\max_{x\in\overline{\Omega}} \left|u(t, x)\right| \leq C \left\|\mathbf{u}(t)\right\|^{\mu}_{L^p(\Omega)}, \quad \forall t \geq 0, \tag{3.1.49}$$

where $u = J\mathbf{u}$. In particular, for every $u \in \widehat{\mathscr{W}^p}(I)$,

$$\lim_{t\to\infty} \max_{x\in\overline{\Omega}} \left|u(t, x)\right| = 0. \tag{3.1.50}$$

Proof. Choose $\mathbf{u} \in \mathscr{W}^p\big([0, \infty)\big)$, and put $u = J\mathbf{u}$. By definition of $\widehat{\mathscr{W}^p}\big([0, \infty)\big)$, $u(t, x) = 0$ whenever $x \in \partial\Omega$. Also, since $p > d$, we find by embedding in a Hölder space (Lemma 3.1.2) that there are constants $M = M(R) > 0$ and $\lambda \in (0, 1)$ such that

$$\left|u(t, x) - u(t, y)\right| \leq M \left|x - y\right|^{\lambda}, \quad \forall x, y \in \overline{\Omega}. \tag{3.1.51}$$

Fix $t > 0$. Let $x_0 \in \overline{\Omega}$ be such that

$$\left|u(t, x_0)\right| = \max_{x\in\overline{\Omega}} \left|u(t, x)\right|. \tag{3.1.52}$$

We are going to estimate $\max_{x\in\overline{\Omega}} \left|u(t,\, x)\right|$ in terms of the L^p norm of $\mathbf{u}(t)$; hence there is nothing to show if $\left|u(t,\, x_0)\right| = 0$. Assuming that $\left|u(t,\, x_0)\right| > 0$, it follows from the Hölder estimate (3.1.51) that for any $x \in \overline{\Omega}$ such that

$$\left|x - x_0\right| < \left(\frac{1}{2M}\left|u(t,\, x_0)\right|\right)^{1/\lambda} =: \delta, \tag{3.1.53}$$

then

$$\left|u(t,\, x) - u(t,\, x_0)\right| \leq \frac{1}{2}\left|u(t,\, x_0)\right|, \tag{3.1.54}$$

whence

$$\left|u(t,\, x)\right| \geq \frac{1}{2}\left|u(t,\, x_0)\right|, \text{ if } \left|x - x_0\right| < \delta. \tag{3.1.55}$$

Notice that because u vanishes on $[0,\, \infty)\times\partial\Omega$, this implies that the ball $B(x_0,\, \delta)$ is contained in Ω. Notice also that $u(t,\, x)$ is of one sign for all $x \in B(x_0,\, \delta)$. From this, we find a lower bound for the L^p norm of $\mathbf{u}(t)$:

$$\begin{aligned}
\left\|\mathbf{u}(t)\right\|^p_{L^p(\Omega)} &\geq \int_{B(x_0,\,\delta)} \left|u(t,\, x)\right|^p \mathrm{d}x && (3.1.56)\\
&\geq \big(\frac{1}{2}\left|u(t,\, x_0)\right|\big)^p C_d \delta^d && (3.1.57)\\
&= 2^{-p}\left|u(t,\, x_0)\right|^p C_d \left(\frac{1}{2M}\left|u(t,\, x_0)\right|\right)^{d/\lambda} && (3.1.58)\\
&= C_{p,d,\lambda,R}\left|u(t,\, x_0)\right|^{p+d/\lambda} && (3.1.59)\\
&= C_{p,d,\lambda,R}\max_{x\in\overline{\Omega}}\left|u(t,\, x)\right|^{p+d/\lambda}, && (3.1.60)
\end{aligned}$$

where C_d is the volume of the unit ball in $\mathbb{R}^d$. The constant

$$C_{p,d,\lambda,R} = 2^{-p}(2M)^{-d/\lambda}C_d > 0 \tag{3.1.61}$$

depends only on p, d, λ, and $M = M(R)$. Hence, using $C = C_{p,d,\lambda,R}^{-\lambda/(\lambda p+d)}$ and $\mu = \lambda p/(\lambda p+d)$, we conclude that

$$\max_{x\in\overline{\Omega}} \left|u(t,\, x)\right| \leq C \left\|\mathbf{u}(t)\right\|_{L^p(\Omega)}^{\mu}, \tag{3.1.62}$$

as desired.

The second claim then follows immediately because functions $\mathbf{u} \in W^{1,p}\big([0,\, \infty),\, L^p(\Omega)\big)$ satisfy

$$\lim_{t\to\infty} \left\|\mathbf{u}(t)\right\|_{L^p(\Omega)} = 0. \tag{3.1.63}$$

$\square$

For later use, we introduce the following subspace of $\mathscr{W}^p(I)$:

$$\mathscr{W}_0^p(I) := W_0^{1,p}(I,\, X^p) \cap L^p(I,\, W^p) \tag{3.1.64}$$

$$= \{\mathbf{u} \in \mathscr{W}^p(I)\colon \mathbf{u}(0) = \mathbf{0}\}. \tag{3.1.65}$$

Note that $\mathscr{W}_0^p(I)$ inherits its norm from $\mathscr{W}^p(I)$. Hence many properties, such as those involving continuity of operators, are inherited by $\mathscr{W}_0^p(I)$. Also note that the image of $\mathscr{W}_0^p(I)$ under the isometry J is

$$\widehat{\mathscr{W}_0^p}(I) := J\mathscr{W}_0^p(I) = \left\{u \in W_0^{1,p}(I \times \Omega)\colon D^\alpha u \in L^p(I \times \Omega) \text{ for all } \left|\alpha\right| = 2\right\}; \tag{3.1.66}$$

recall that the notation D^α is reserved for derivatives with respect to $x-$variables.

3.2 SMOOTHNESS OF THE NEMYTSKII OPERATORS

3.2.1 Linear and nonlinear Nemytskii operations

We begin with the following two lemmas. The first lemma shows that a path of bounded linear operators on W^p can be viewed as a linear operator on $\mathscr{W}^p(I)$, and hence (via J) as a linear operator on $\widehat{\mathscr{W}^p}(I)$. The second lemma draws a similar conclusion for nonlinear functions $G = G(t, \xi)$.

Lemma 3.2.1 (Linear Nemytskii Operator). *Let $A : I \to \mathcal{L}\big(W^p, X^p\big)$ be continuous and bounded on I, and also suppose that A is uniformly continuous on bounded subsets of I. Let $\mathbf{u} \in \mathscr{W}^p(I)$. Then the relation*

$$\mathbf{v}(t) := A(t)\mathbf{u}(t), \quad \textit{(pointwise multiplication)} \tag{3.2.1}$$

defines an element $\mathbf{v} \in \mathscr{X}^p(I)$, and

$$\big\|\mathbf{v}\big\|_{\mathscr{X}^p(I)} \leq \left(\sup_{t\in I} \big\|A(t)\big\|_{op}\right) \big\|\mathbf{u}\big\|_{\mathscr{W}^p(I)} . \tag{3.2.2}$$

Proof. First, we settle the question of measurability of $\mathbf{v}$, which is well-defined pointwise almost everywhere by the relation (3.2.1). We proceed directly, by showing that $\mathbf{v}$ is the limit in Lebesgue measure μ of a sequence of simple functions $(\mathbf{v}_n)$. To construct this sequence, we begin (by definition of measurability) with a sequence $(\mathbf{u}_n)$ of simple functions that converges in measure to $\mathbf{u}$. That $\mathbf{u}_n$ is simple means that for each $n \in \mathbb{N}$ there are finitely many $\mathbf{u}_{nk} \in X^p$ and corresponding measurable subsets E_{nk} of I such that

$$\mathbf{u}_n = \sum_k \mathbf{u}_{nk}\chi_{E_{nk}}, \quad \text{(finite sum)} \tag{3.2.3}$$

where $\chi_{E_{nk}}$ is the characteristic function associated to the set E_{nk}:

$$\chi_{E_{nk}}(t) := \begin{cases} 1 & \text{if } t \in E_{nk}, \\ \\ 0 & \text{if } t \notin E_{nk}. \end{cases} \tag{3.2.4}$$

We will also use some simple functions to approximate A. For each $n \in \mathbb{N}$, let $I_n = I \cap [-n, n]$, which may be empty. For sufficiently large n, I_n is not empty, and we partition I_n into finitely many nonempty pairwise disjoint subintervals I_{nk}, each of length less than 2^{-n}. Choose points $t_{nk} \in I_{nk}$, say the midpoints.

We define, for each $t \in I$ and each $n \in \mathbb{N}$,

$$A_n(t) := \begin{cases} A(t_{nk}) & \text{if } t \in I_{nk}, \\ \\ 0 & \text{if } t \notin I_n. \end{cases} \tag{3.2.5}$$

Hence, A_n converges to A uniformly on bounded subsets of I, because of the assumed uniform continuity of A on the bounded subsets of I and the convergence of the diameters of the subintervals I_{nk} to zero as $n \to \infty$. We then set

$$\mathbf{v}_n(t) := A_n(t)\mathbf{u}_n(t) \in X^p. \tag{3.2.6}$$

Then each $\mathbf{v}_n$ is a measurable simple function, because

$$\mathbf{v}_n = \sum_{j,\,k} A_{nk}\mathbf{u}_{nj}\chi_{I_{nk}\cap E_{nj}}. \tag{3.2.7}$$

According to Corollary III.6.13 in Dunford and Schwartz [DS88], by passing to a subsequence, we may assume that the sequence $\mathbf{u}_n$ converges pointwise almost everywhere to $\mathbf{u}$. Hence, the sequence $\mathbf{v}_n$ converges almost everywhere to $\mathbf{v}$. By Corollary III.6.14 in [DS88], $\mathbf{v}$ is therefore measurable.

To complete the proof, we need only point out that

$$\int_I \left\|A(t)\mathbf{u}(t)\right\|_{X^p}^p \mathrm{d}t \leq \int_I \left(\sup_{s\in I}\left\|A(s)\right\|\right)^p \left\|\mathbf{u}(t)\right\|_{W^p}^p \mathrm{d}t \tag{3.2.8}$$

$$= \left(\sup_{t\in I}\left\|A(t)\right\|\right)^p \left\|\mathbf{u}\right\|_{\mathscr{W}^p(I)}^p . \tag{3.2.9}$$

□

The above lemma justifies the following definition.

Definition 3.2.2 (Linear Nemytskii Operators). Let $A : I \to \mathcal{L}\big(W^p,\, X^p\big)$ be continuous and bounded on I. We define a continuous linear Nemytskii operator

$$A^\sharp \in \mathcal{L}\big(\mathscr{W}^p(I),\, \mathscr{X}^p(I)\big) \tag{3.2.10}$$

by setting

$$(A^\sharp\mathbf{u})(t) := A(t)\mathbf{u}(t), \quad \forall t \in I, \forall \mathbf{u} \in \mathscr{W}^p(I). \tag{3.2.11}$$

(pointwise multiplication)

Accordingly, we define a continuous linear Nemytskii operator

$$\widetilde{A} \in \mathcal{L}\big(\widehat{\mathscr{W}^p}(I),\, \widehat{\mathscr{X}^p}(I)\big) \tag{3.2.12}$$

by setting

$$\widetilde{A} := JA^\sharp J^{-1}. \tag{3.2.13}$$

We now incorporate some nonlinearities into our framework.

Lemma 3.2.3 (Nonlinear Nemytskii Operator). *Let $G = G(t,\xi) \in C^0(I \times \mathbb{R}, \mathbb{R})$ be such that $D_\xi G$ exists and is bounded on $I \times K$ for each bounded interval $K \subset \mathbb{R}$, and such that $G(t, 0) = 0$ for all $t \in I$. Let $u \in \widehat{\mathscr{W}^p}(I)$, and define*

$$v(t, x) := G\big(t, u(t, x)\big). \tag{3.2.14}$$

Then $v \in \widehat{\mathscr{X}^p}(I)$.

Proof. The measurability of v is clear from the measurability of u and the continuity of G. To show that $v \in \widehat{\mathscr{X}^p}(I)$, begin with a bounded interval K that contains the range of u. The existence of K follows from the embedding of Lemma 3.1.2 with the decay property of Lemma 3.1.17. We then obtain by assumption a bound $M > 0$ for the values of $D_\xi G$ on $I \times K$. As a result, for each $(t, x) \in I \times \Omega$

$$\begin{aligned} \left|G\big(t, u(t, x)\big)\right| &= \left| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} G\big(t, su(t, x)\big)\,\mathrm{d}s \right| && (3.2.15)\\ &\le \left| \int_0^1 D_\xi G\big(t, su(t, x)\big)\,\mathrm{d}s \right| \left|u(t, x)\right| && (3.2.16)\\ &\le M \left|u(t, x)\right|. && (3.2.17) \end{aligned}$$

Hence, upon taking p^{th} powers and integrating,

$$\int_{I\times\Omega} \left|v(t, x)\right|^p \mathrm{d}(t, x) \le M^p \left\|u\right\|_{\widehat{\mathscr{X}^p}(I)} < \infty, \tag{3.2.18}$$

which proves that $v \in \widehat{\mathscr{X}^p}(I)$. $\square$

Lemma 3.2.3 justifies the following definition.

Definition 3.2.4 (Nonlinear Nemytskii Operators). Let $G = G(t, \xi) \in C^0(I \times \mathbb{R}, \mathbb{R})$ be such that $D_\xi G$ exists and is bounded on $I \times K$ for each bounded interval K and such that $G(t, 0) = 0$ for all $t \in I$. We define a Nemytskii operator $\widetilde{G} : \widehat{\mathscr{W}^p}(I) \to \widehat{\mathscr{X}^p}(I)$ as follows. For each $u \in \widehat{\mathscr{W}^p}(I)$,

$$\widetilde{G}(u)(t, x) := G\big(t, u(t, x)\big), \quad \forall (t, x) \in I \times \Omega. \tag{3.2.19}$$

Using the isometry J, we accordingly define a Nemytskii operator $G^\sharp : \mathscr{W}^p(I) \to \mathscr{X}^p(I)$ by

$$G^\sharp := J^{-1} \circ \widetilde{G} \circ J. \tag{3.2.20}$$

Remark 3.2.5. Once again, we note the contrast between the convenience of the explicit structure of $\mathscr{W}^p(I)$ in developing the linear Nemytskii operator, and the convenience of the explicit structure of $\widehat{\mathscr{W}^p}(I)$ in developing the nonlinear Nemytskii operator. $\Diamond$

3.2.2 A smooth operator

Suppose now that we are given a family $\big(A(t)\big)_{t \in I}$ of bounded linear operators from $W^{2,p}(\Omega)$ into $L^p(\Omega)$. We assume that the following conditions hold for $A = A(t)$:

A is bounded on I; (3.2.21a)

A is uniformly continuous on the bounded subsets of I; (3.2.21b)

For some $A^\infty \in \mathcal{L}\big(W^{2,p}(\Omega), L^p(\Omega)\big)$, one has $\lim_{t \to \infty} \big\|A(t) - A^\infty\big\|_{\mathrm{op}} = 0$. (3.2.21c)

Suppose also that we are given a function $G = G(t,\, \xi)\colon I \times \mathbb{R} \to \mathbb{R}$ that satisfies the following conditions:

$$G = G(t,\, \xi) \in C^0(I \times \mathbb{R},\, \mathbb{R}) \text{ and the partial derivative } D_\xi G \text{ exists;} \tag{3.2.22a}$$

$$G \text{ and } D_\xi G \text{ are bounded and uniformly continuous on } I \times K, \text{ for all bounded } K \subset \mathbb{R}; \tag{3.2.22b}$$

$$G(t,\, 0) = D_\xi G(t,\, 0) = 0, \text{ for all } t \in I. \tag{3.2.22c}$$

Of course, condition (3.2.22b) is without content if I is compact, but we are primarily interested in $I = [0,\, \infty)$. We recall the standing assumption that $p > d+1$, so that by Lemma 3.1.2, $\widehat{\mathscr{W}^p}(I)$ is continuously embedded in $C^{0,\,\lambda}(\overline{I \times \Omega})$ for some $\lambda \in (0,\, 1)$. According to the results of Lemmas 3.2.1 and 3.2.3, the following is a well defined operator from $\mathscr{W}^p(I)$ into $\mathscr{X}^p(I)$:

$$\Phi_{A,\,G}(\mathbf{u}) := \left(\frac{\mathrm{d}}{\mathrm{d}t} - A^\sharp\right)\mathbf{u} + G^\sharp(\mathbf{u}), \quad \forall \mathbf{u} \in \mathscr{W}^p(I). \tag{3.2.23}$$

It is the primary purpose of this section to demonstrate that $\Phi_{A,\,G}$ is continuously differentiable, and to find an expression for $D\Phi_{A,\,G}$. We restrict our attention to the term $G^\sharp$ because according to Lemma 3.2.1, $\frac{\mathrm{d}}{\mathrm{d}t} - A^\sharp$ is continuous and linear.

Lemma 3.2.6 (Continuity). *Assume that G satisfies conditions (3.2.22a), (3.2.22b), and (3.2.22c). The operator $G^\sharp : \mathscr{W}^p(I) \to \mathscr{X}^p(I)$ is continuous and weakly sequentially continuous*[1].

Proof. Let $\mathbf{u}$ and $\mathbf{v}$ be elements of $\mathscr{W}^p(I)$. Set $u = J\mathbf{u}$ and $v = J\mathbf{v}$, and let K be a bounded interval that contains the ranges of both u and v. Let M be a bound for $D_\xi G$ on $I \times K$. We

[1] meaning that weakly convergent sequences are mapped to weakly convergent sequences.

estimate that for all $(t,\, x) \in I \times \Omega$,

$$
\begin{aligned}
\big|G\big(t,\, u(t,\, x)\big) \; &-G\big(t,\, v(t,\, x)\big)\big| \\
&= \left| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} G\big(t,\, su(t,\, x) + (1-s)v(t,\, x)\big)\, \mathrm{d}s \right| && (3.2.24) \\
&\leq \left| \int_0^1 D_\xi G\big(t,\, su(t,\, x) + (1-s)v(t,\, x)\big)\, \mathrm{d}s \right| \big|u(t,\, x) - v(t,\, x)\big| && (3.2.25) \\
&\leq M \,\big|u(t,\, x) - v(t,\, x)\big| \,. && (3.2.26)
\end{aligned}
$$

We take p^{th} powers, and we integrate with respect to $(t,\, x)$ on $I \times \Omega$:

$$
\left\| \widetilde{G}(u) - \widetilde{G}(v) \right\|^p_{\widehat{\mathscr{X}^p}(I)} \leq M \left\| u - v \right\|^p_{\widehat{\mathscr{X}^p}(I)} . \tag{3.2.27}
$$

Because of the isometry J between $\mathscr{X}^p(I)$ and $\widehat{\mathscr{X}^p}(I)$, this shows that

$$
\left\| G^\sharp(\mathbf{u}) - G^\sharp(\mathbf{v}) \right\|_{\mathscr{X}^p(I)} \leq M^{1/p} \left\| \mathbf{u} - \mathbf{v} \right\|_{\mathscr{X}^p(I)} . \tag{3.2.28}
$$

This establishes the continuity of $G^\sharp$.

For the weak sequential continuity, suppose that $(\mathbf{u}_n)$ converges weakly in $\mathscr{X}^p(I)$, to some $\mathbf{u} \in \mathscr{X}^p(I)$. As in the preceding part of the proof, it is convenient to work in $\widehat{\mathscr{W}^p}(I)$; put $(u_n) := (J\mathbf{u}_n)$ and $u := J\mathbf{u}$. Since J is an isometry of Banach spaces, both J and J^{-1} are weakly sequentially continuous. Hence, the sequence (u_n) is weakly convergent in $\widehat{\mathscr{W}^p}(I)$ to u, and we wish to show that $\big(\widetilde{G}(u_n)\big)$ is weakly convergent to $\widetilde{G}(u)$ in $\widehat{\mathscr{X}^p}(I)$.

We first show that (u_n) converges uniformly to u on compact subsets of $I \times \Omega$. The sequence (u_n) is weakly convergent in $\widehat{\mathscr{W}^p}(I)$, and hence is bounded in $\widehat{\mathscr{W}^p}(I)$. Because of the Hölder embedding of $\widehat{\mathscr{W}^p}(I)$ into $C^{0,\lambda}(\overline{I \times \Omega})$ (Lemma 3.1.2), the sequence (u_n) is also bounded in $C^{0,\lambda}$. There is therefore a constant $M \geq 0$ such that

$$
\big|u_n(t,\, x) - u_n(t',\, x')\big| \leq M \left(\big|t - t'\big| + \big|x - x'\big|\right)^\lambda . \tag{3.2.29}
$$

In particular, the sequence (u_n) is equicontinuous. Fix a compact subset K of $I \times \Omega$. Let $\epsilon > 0$. From the equicontinuity of (u_n) (and the continuity of u) we obtain $\delta > 0$ such that

$$\left|u_n(t,\, x) - u_n(t',\, x')\right| + \left|u(t,\, x) - u(t',\, x')\right| < \epsilon/2 \tag{3.2.30}$$

whenever $n \in \mathbb{N}$ and $(t,\, x)$ and $(t',\, x')$ are δ-close points in K. Cover K with finitely many δ balls, centered at points $(t_1,\, x_1)$,...$(t_N,\, x_N)$ in K. Hence, for any $(t,\, x) \in K$, and $n \in \mathbb{N}$, we have the estimate

$$\begin{aligned} \left|u_n(t,\, x) - u(t,\, x)\right| &\leq \left|u_n(t,\, x) - u_n(t_k,\, x_k)\right| \\ &\quad + \left|u_n(t_k,\, x_k) - u(t_k,\, x_k)\right| + \left|u(t_k,\, x_k) - u(t,\, x)\right| \\ &< \epsilon/2 + \left|u_n(t_k,\, x_k) - u(t_k,\, x_k)\right|, \end{aligned} \tag{3.2.31}$$

where $(t_k,\, x_k)$ is the center of a δ-ball that contains $(t,\, x)$. Now, because (u_n) also converges weakly to u in $W^{1,p}(I \times \Omega)$, we know from Lemma 8 (ii) of Rabier [Rab04a] that (u_n) converges to u pointwise. Hence, for sufficiently large $n \in \mathbb{N}$, we will have

$$\left|u_n(t_k,\, x_k) - u(t_k,\, x_k)\right| < \epsilon/2, \tag{3.2.32}$$

independently of our choice of $k = 1,\, \ldots,\, N$. Altogether,

$$\left|u_n(t,\, x) - u(t,\, x)\right| < \epsilon \tag{3.2.33}$$

for sufficiently large $n \in \mathbb{N}$, independently of the choice of $(t,\, x) \in K$. This shows that (u_n) converges to u uniformly on compact sets.

To establish the weak convergence of $\widetilde{G}(u_n)$ to $\widetilde{G}(u)$ in $\widehat{\mathscr{X}^p}(I) = L^p(I \times \Omega)$, it is sufficient[2] to show that

$$\int_{I\times\Omega} \Big(G\big(t,\, u_n(t,\, x)\big) - G\big(t,\, u(t,\, x)\big) \Big) \phi(t,\, x) \,\mathrm{d}(t,\, x) \to 0 \text{ as } n \to \infty \tag{3.2.34}$$

for all $\phi \in C_0^\infty\big(I \times \Omega\big)$. Let K be a compact interval that contains the ranges of all of the functions (u_n) and u. Because of the uniform convergence of (u_n) to u on the compact support of ϕ, and the uniform continuity of G on $I \times K$, (3.2.34) holds, and the proof is complete. □

Corollary 3.2.7 (Continuity). *Assume that G satisfies conditions (3.2.22a), (3.2.22b), and (3.2.22c). The operator $\widetilde{G} : \widehat{\mathscr{W}^p}(I) \to \widehat{\mathscr{X}^p}(I)$ is continuous and weakly sequentially continuous.*

Lemma 3.2.8 (Differentiability). *Assume that G satisfies conditions (3.2.22a), (3.2.22b), and (3.2.22c). The operator $\widetilde{G}$ is differentiable from $\widehat{\mathscr{W}^p}(I)$ into $\widehat{\mathscr{X}^p}(I)$. For each $u \in \widehat{\mathscr{W}^p}(I)$, the derivative of $\widetilde{G}$ at u is given by*

$$\big(D\widetilde{G}(u)h\big)(t,\, x) = D_\xi G\big((t,\, u(t,\, x)\big)h(t,\, x), \quad \forall h \in \widehat{\mathscr{W}^p}(I), \forall (t,\, x) \in I \times \Omega. \tag{3.2.35}$$

Proof. Let $\epsilon > 0$ be given. For later convenience, we denote

$$v(t,\, x) = D_\xi G\big((t,\, u(t,\, x)\big)h(t,\, x), \quad \forall (t,\, x) \in I \times \Omega. \tag{3.2.36}$$

[2]The space $C_0^\infty\big(I \times \Omega\big)$ is dense in $L^{p'}(I \times \Omega)$, which represents $\big(\widehat{\mathscr{X}^p}(I)\big)^*$.

Because $\operatorname{rge} u$ is a bounded interval K, and $D_\xi G$ is bounded on $I \times K$, it is true that the map carrying $h \in \widehat{\mathscr{W}^p}(I)$ to v is indeed a bounded linear map from $\widehat{\mathscr{W}^p}(I)$ into $\widehat{\mathscr{X}^p}(I)$. It remains to verify that $h \mapsto v$ is in fact the derivative of $\widetilde{G}$ at the point u. For all $(t, x) \in I \times \Omega$,

$$G\big(t, u(t, x) + h(t, x)\big) - G\big(t, u(t, x)\big) - v(t, x)$$

$$= \Big(\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} G\big(t, u(t, x) + sh(t, x)\big)\, \mathrm{d}s\Big) - D_\xi G\big(t, u(t, x)\big) h(t, x) \tag{3.2.37}$$

$$= \Big(\int_0^1 D_\xi G\big(t, u(t, x) + sh(t, x)\big) - D_\xi G\big(t, u(t, x)\big)\, \mathrm{d}s\Big) h(t, x). \tag{3.2.38}$$

Let K be a bounded interval large enough to contain the range of any function in $\widehat{\mathscr{W}^p}(I)$ of norm no larger than $\|u\|_{\widehat{\mathscr{W}^p}(I)} + 1$. Using the uniform continuity of $D_\xi G$ on $I \times K$, there is some $\delta > 0$ such that

$$\big|D_\xi G\big(t, u(t, x) + sh(t, x)\big) - D_\xi G\big(t, u(t, x)\big)\big| < \epsilon \tag{3.2.39}$$

as long as

$$\big|h(t, x)\big| < \min(1, \delta). \tag{3.2.40}$$

Thus, using the embedding of $\widehat{\mathscr{W}^p}(I)$ in $C^0(\overline{I \times \Omega})$ (see Lemma 3.1.2), for sufficiently small $\|h\|_{\widehat{\mathscr{W}^p}(I)}$ we have $\|h\|_\infty < \min(1, \delta)$ so that altogether

$$\big|G\big(t, u(t, x) + h(t, x)\big) - G\big(t, u(t, x)\big) - D_\xi G\big(t, u(t, x)\big) h(t, x)\big| \le \epsilon \big|h(t, x)\big|. \tag{3.2.41}$$

We take p^{th} powers and integrate; this results in the estimate

$$\Big\|\widetilde{G}(u + h) - \widetilde{G}(u) - v\Big\|_{\widehat{\mathscr{X}^p}(I)} \le \epsilon \|h\|_{\widehat{\mathscr{X}^p}(I)}. \tag{3.2.42}$$

Since $\|h\|_{\widehat{\mathscr{X}^p}(I)} \le \|h\|_{\widehat{\mathscr{W}^p}(I)}$, we have shown that for all sufficiently small $\|h\|_{\widehat{\mathscr{W}^p}(I)} \neq 0$,

$$\frac{\Big\|\widetilde{G}(u + h) - \widetilde{G}(u) - v\Big\|_{\widehat{\mathscr{X}^p}(I)}}{\|h\|_{\widehat{\mathscr{W}^p}(I)}} \le \epsilon, \tag{3.2.43}$$

which proves the desired result that $D\widetilde{G}(u)h = v$. $\square$

Corollary 3.2.9. *Assuming that G satisfies conditions (3.2.22a), (3.2.22b), and (3.2.22c), the Nemytskii operator $\widetilde{D_\xi G}\colon \widehat{\mathscr{W}^p}(I) \to L^\infty(I \times \Omega)$ is well defined by the relation*

$$\widetilde{D_\xi G}(u)(t,\, x) = D_\xi G\big(t,\, u(t,\, x)\big). \tag{3.2.44}$$

Proof. This simple result is not exactly a corollary of Lemma 3.2.8, but was noticed in the opening of the proof of Lemma 3.2.8. □

Corollary 3.2.10. *Assuming that G satisfies conditions (3.2.22a), (3.2.22b), and (3.2.22c), the Nemytskii operator $(D_\xi G)^\sharp\colon \mathscr{W}^p(I) \to L^\infty\big(I,\, L^\infty(\Omega)\big)$ is well defined by the relation*

$$(D_\xi G)^\sharp(\mathbf{u}) = J^{-1}\widetilde{D_\xi G}(J\mathbf{u}). \tag{3.2.45}$$

If we put $u = J\mathbf{u}$ and $h = J\mathbf{h}$, this may be expressed as

$$(D_\xi G)^\sharp(\mathbf{u})\mathbf{h} = D_\xi G\big(\cdot,\, u(\cdot,\, \cdot)\big)h(\cdot,\, \cdot). \tag{3.2.46}$$

Proof. According to the preceding corollary, $\widetilde{D_\xi G}(J\mathbf{u})$ is a measurable, essentially bounded function on $I \times \Omega$. Hence, the only potential difficulty with identifying the partial map $t \mapsto D_\xi G\big(t,\, u(t,\, \cdot)\big)$ with an element $J^{-1}\widetilde{D_\xi G}(J\mathbf{u})$ of $L^\infty\big(I,\, L^\infty(\Omega)\big)$ is the issue of measurability. If I is bounded, then $L^\infty \subset L^1$, and we can appeal to part (b) of Lemma III.11.16 of Dunford and Schwartz [DS88]. Otherwise, we do so with the bounded subsets of I. Measurability on I then follows, for example, by Theorem III.6.10 of Dunford and Schwartz [DS88]. □

Remark 3.2.11. The value of Corollary 3.2.10 is mostly notational; the right hand sides of both equations (3.2.45) and (3.2.46) are rather awkward. For example, the reader might try to formulate the next result without the use of the notation $(D_\xi G)^\sharp$, which is otherwise undefined.

Corollary 3.2.12 (Differentiability). *Assume that G satisfies (3.2.22a), (3.2.22b), and (3.2.22c). The operator $G^\sharp$ is differentiable from $\mathscr{W}^p(I)$ into $\mathscr{X}^p(I)$, and $DG^\sharp = (D_\xi G)^\sharp$.*

Proof. Let $\mathbf{u}$ and $\mathbf{h}$ be in $\mathscr{W}^p(I)$, and let $u = J\mathbf{u}$ and $h = J\mathbf{h}$. Since J in an isometry,

$$\left\|G^\sharp(\mathbf{u}+\mathbf{h}) - G^\sharp(\mathbf{u}) - (D_\xi G)^\sharp(\mathbf{u})\mathbf{h}\right\|_{\mathscr{X}^p} = \left\|\widetilde{G}(u+h) - \widetilde{G}(u) - \widetilde{D_\xi G}(u)h\right\|_{\widehat{\mathscr{X}^p}}, \tag{3.2.47}$$

where we have used equation (3.2.45) in Corollary 3.2.10. With use of equation (3.2.44) in Corollary 3.2.9 and equation (3.2.35) in Lemma 3.2.8, this becomes

$$\left\|G^\sharp(\mathbf{u}+\mathbf{h}) - G^\sharp(\mathbf{u}) - (D_\xi G)^\sharp(\mathbf{u})\mathbf{h}\right\|_{\mathscr{X}^p} = \left\|\widetilde{G}(u+h) - \widetilde{G}(u) - D\widetilde{G}(u)h\right\|_{\widehat{\mathscr{X}^p}}. \tag{3.2.48}$$

Since $\left\|\mathbf{h}\right\|_{\mathscr{W}^p} = \left\|h\right\|_{\widehat{\mathscr{W}^p}}$, this proves that $DG^\sharp(u) = (D_\xi G)^\sharp(u)$. □

Lemma 3.2.13 ($\mathbf{C^1}$). *Assume that G satisfies conditions (3.2.22a) – (3.2.22c). The map $D\widetilde{G} : \widehat{\mathscr{W}^p}(I) \to \mathcal{L}\big(\widehat{\mathscr{W}^p}(I),\, \widehat{\mathscr{X}^p}(I)\big)$ is continuous.*

Proof. Fix $u \in \widehat{\mathscr{W}^p}(I)$ and let $\epsilon > 0$. Let K be a bounded interval containing the ranges of u and of all $v \in \widehat{\mathscr{W}^p}(I)$ such that $\left\|u - v\right\|_{\widehat{\mathscr{W}^p}(I)} < 1$. Using the uniform continuity of $D_\xi G$ on $I \times K$, there is $\delta > 0$ such that if $\left\|u - v\right\|_{\widehat{\mathscr{W}^p}(I)} < \min(\delta,\, 1)$ then

$$\left|D_\xi G\big(t,\, u(t,\, x)\big) - D_\xi G\big(t,\, v(t,\, x)\big)\right| < \epsilon, \quad \forall (t,\, x) \in I \times \Omega. \tag{3.2.49}$$

Hence, if $h \in \widehat{\mathscr{W}^p}(I)$ is such that $\left\|h\right\|_{\widehat{\mathscr{W}^p}(I)} \leq 1$, then

$$\begin{aligned}
&\left\|(D\widetilde{G}(u) - D\widetilde{G}(v))h\right\|^p_{\widehat{\mathscr{X}^p}} \\
&\qquad = \int_{I\times\Omega} \left|\Big(D_\xi G\big(t,\, u(t,\, x)\big) - D_\xi G\big(t,\, v(t,\, x)\big)\Big)h(t,\, x)\right|^p \mathrm{d}(t,\, x) && (3.2.50)\\
&\qquad \leq \epsilon^p \int_{I\times\Omega} \left|h(t,\, x)\right|^p \mathrm{d}(t,\, x) && (3.2.51)\\
&\qquad = \epsilon^p \left\|h\right\|^p_{\widehat{\mathscr{X}^p}(I)} && (3.2.52)\\
&\qquad \leq \epsilon^p. && (3.2.53)
\end{aligned}$$

This shows that whenever v is such that $\|u - v\|_{\widehat{\mathscr{W}}^p(I)} < \min(\delta, 1)$, then the norm of $D\widetilde{G}(u) - D\widetilde{G}(v)$ in $\mathcal{L}(\widehat{\mathscr{W}}^p(I), \widehat{\mathscr{X}}^p(I))$ is less than ϵ. This shows the desired continuity at u. □

Corollary 3.2.14 ($\mathbf{C^1}$). *Assume that G satisfies conditions (3.2.22a) – (3.2.22c). The map $DG^\sharp : \mathscr{W}^p(I) \to \mathcal{L}\big(\mathscr{W}^p(I), \mathscr{X}^p(I)\big)$ is continuous.*

Proof. Continuity follow directly from Lemma 3.2.13, because J is an isometry. □

To summarize, we have proved the following theorem:

Theorem 3.2.15. *Assume that G satisfies conditions (3.2.22a), (3.2.22b), and (3.2.22c). The operator $\Phi_{A,G}$ defined in (3.2.23) is a C^1 map from $\mathscr{W}^p(I)$ into $\mathscr{X}^p(I)$. The derivative of this operator satisfies*

$$\big(D\Phi_{A,G}(\mathbf{u})\big)\mathbf{h} = \dot{\mathbf{h}} - A^\sharp\mathbf{h} + (D_\xi G)^\sharp(\mathbf{u})\mathbf{h}. \tag{3.2.54}$$

Equivalently, if $u = J\mathbf{u}$ and $h = J\mathbf{h}$,

$$\big(DJ\Phi_{A,G}(\mathbf{u})\big)h = \frac{\partial h}{\partial t} - \widetilde{A}h + \widetilde{D_\xi G}(u)h. \tag{3.2.55}$$

We are primarily interested not in $\Phi_{A,G}$ itself, but rather in an augmentation of $\Phi_{A,G}$ that includes information about $u(0, \cdot)$. Now is a good time to notice that evaluation at $t = 0$ is a continuous linear map:

Lemma 3.2.16. *For $\mathbf{u} \in \mathscr{W}^p\big([0, \infty)\big)$, put*

$$E_0(\mathbf{u}) := \mathbf{u}(0) \in L^p(\Omega). \tag{3.2.56}$$

Then E_0 is a well defined continuous linear map of $\mathscr{W}^p\big([0, \infty)\big)$ into $X^p = L^p(\Omega)$.

Proof. Linearity is obvious, and functions in the Sobolev space $W^{1,p}\big([0, \infty), L^p(\Omega)\big)$ are continuous from $[0, \infty)$ to $L^p(\Omega)$. □

3.3 THE FREDHOLM PROPERTY AND INDEX

In order to make eventual use of the available degree theory, we wish to find conditions to ensure that $\Phi_{A,G}$ is Fredholm of index 0. Recall (see Section 1.4) that this means that for all $\mathbf{u} \in \mathscr{W}^p(I)$, the linear map $D\Phi_{A,G}(\mathbf{u}) \in \mathcal{L}(\mathscr{W}^p(I),\ \mathscr{X}^p(I))$ is Fredholm of index 0, which in turn means that

$$\dim\ker D\Phi_{A,G}(\mathbf{u}) < \infty, \tag{3.3.1}$$

$$\operatorname{codim}\operatorname{rge} D\Phi_{A,G}(\mathbf{u}) < \infty, \tag{3.3.2}$$

and that the Fredholm index $\dim\ker D\Phi_{A,G}(\mathbf{u}) - \operatorname{codim}\operatorname{rge} D\Phi_{A,G}(\mathbf{u})$ is zero. We will show in Lemma 3.3.1 that $D\Phi_{A,G}(\mathbf{0}) - D\Phi_{A,G}(\mathbf{u})$ is compact for all $\mathbf{u} \in \mathscr{W}^p(I)$. The Fredholm property for $\Phi_{A,G}$ will then follow from that of $D\Phi_{A,G}(\mathbf{0})$, if available. This leads us to study when $D\Phi_{A,G}(\mathbf{0}) = \frac{\mathrm{d}}{\mathrm{d}t} - A^\sharp(0)$ is Fredholm. This will bring us back to the fact that this property depends on the choice source and target spaces, and we will replace $\Phi_{A,G}$ with its restriction to $\mathscr{W}_0^p(I)$. (The target space will remain $\mathscr{X}^p(I)$.) After this, we will study the Fredholm property of the map with evaluation $\big(\Phi_{A,G},\ E_0\big)$. As we shall see, the correct functional setting is obtained when $\Phi_{A,G}$ is again viewed as a map on $\mathscr{W}^p(I)$, and the target space of E_0 is taken to be $\operatorname{rge} E_0$.

Lemma 3.3.1 (Compact Perturbation). *Let $\mathbf{u} \in \mathscr{W}^p(I)$. Suppose that A and G satisfy the conditions (3.2.21) and (3.2.22) on page 133. Then $D\Phi_{A,G}(\mathbf{u}) - D\Phi_{A,G}(\mathbf{0})$ is a compact linear operator from $\mathscr{W}^p(I)$ into $\mathscr{X}^p(I)$.*

Proof. Let $(\mathbf{h}_n)$ be a bounded sequence in $\mathscr{W}^p(I)$.

We work instead in $\widehat{\mathscr{W}^p}(I)$. We hence take $u := J\mathbf{u}$, $h_n := J\mathbf{h}_n$,

$$B := J\big(D\Phi_{A,G}(\mathbf{u}) - D\Phi_{A,G}(\mathbf{0})\big)J^{-1} = \widetilde{D_\xi G}(u), \tag{3.3.3}$$

and $g_n := Bh_n$ so that

$$g_n(t,x) = D_\xi G\big(t, u(t,x)\big)h_n(t,x). \tag{3.3.4}$$

Note that it is sufficient to show that the sequence $(g_n) := (Bh_n)$ is relatively compact in $\widehat{\mathscr{X}^p}(I)$. We proceed by first verifying that B is continuous as a map from $\widehat{\mathscr{X}^p}(I)$ into itself (not just from $\widehat{\mathscr{W}^p}(I)$ into $\widehat{\mathscr{X}^p}(I)$). Then, we check that (h_n) has a Cauchy subsequence with respect to the norm of $\widehat{\mathscr{X}^p}(I')$, if I' is any bounded subinterval of I. The relative compactness of (g_n) then follows immediately in the case that I is bounded. Otherwise, we use decay of $D_\xi G\big(t, u(t,x)\big)$ as $|t| \to \infty$ to achieve the desired Cauchy property for the sequence (g_n).

For the continuity of B on $\widehat{\mathscr{X}^p}(I)$, let $M > 0$ be a bound for $D_\xi G$ on $I \times K$, where K is a compact interval containing the range of the bounded function u. Then, for any $v \in \widehat{\mathscr{X}^p}(I) = L^p(I \times \Omega)$:

$$\int_{I\times\Omega} \big|(Bv)(t,x)\big|^p \,\mathrm{d}(t,x) = \int_{I\times\Omega} \big|D_\xi G\big(t, u(t,x)\big)v(t,x)\big|^p \,\mathrm{d}(t,x) \tag{3.3.5}$$

$$\leq M^p \int_{I\times\Omega} \big|v(t,x)\big|^p \,\mathrm{d}(t,x). \tag{3.3.6}$$

This shows the continuity of B on $\widehat{\mathscr{X}^p}(I)$. Next, consider any bounded subinterval I' of I. The compact embedding of $\mathscr{W}^p(I')$ in $\mathscr{X}^p(I')$ is a result of Theorem 1 of Simon [Sim87]. Thus, (h_n) has an $\widehat{\mathscr{X}^p}(I')$-convergent subsequence (h_{n_k}). Since $(g_{n_k}) = (Bh_{n_k})$ is then convergent, the proof is complete in case I is itself bounded.

Now suppose that I is unbounded, and let $\epsilon > 0$. According to Lemma 3.1.17, there is a bounded subinterval I' of I such that

$$\big|u(t,\, x)\big| < \epsilon, \quad \forall (t,\, x) \in (I \setminus I') \times \Omega. \tag{3.3.7}$$

Further, according to the uniform continuity expressed in (3.2.22b), and assumption (3.2.22c) that $D_\xi G(t,\, 0) = 0$ for all t, we can ensure that

$$\big|D_\xi G\big(t,\, u(t,\, x)\big)\big| < \epsilon, \quad \forall (t,\, x) \in (I \setminus I') \times \Omega, \tag{3.3.8}$$

by possibly enlarging I'.

According to the compact embedding of $\mathscr{W}^p(I')$ in $\mathscr{X}^p(I')$, by passing to a subsequence we may suppose that the restrictions to I' of the functions $\mathbf{h}_n$ form a Cauchy sequence in $\mathscr{X}^p(I')$. It then follows from the isometry of $\mathscr{X}^p(I')$ with $\widehat{\mathscr{X}^p}(I')$ and from the continuity of B that the sequence of restrictions of the functions g_n to $I' \times \Omega$ is Cauchy in $\widehat{\mathscr{X}^p}(I')$. Hence, there is some $N \in \mathbb{N}$ such that

$$\int_{I' \times \Omega} \Big|D_\xi G\big(t,\, u(t,\, x)\big)\Big(h_n(t,\, x) - h_m(t,\, x)\Big)\Big|^p \,\mathrm{d}(t,\, x) < \epsilon \tag{3.3.9}$$

for all $n,\, m > N$.

Finally, by the estimate (3.3.8) that resulted from the choice of I', we have

$$\int_{(I \setminus I') \times \Omega} \Big|D_\xi G\big(t,\, u(t,\, x)\big)\Big(h_n(t,\, x) - h_m(t,\, x)\Big)\Big|^p \,\mathrm{d}(t,\, x) < \epsilon^p M^p, \tag{3.3.10}$$

where M is a bound for the sequence (h_n) in $\widehat{\mathscr{X}^p}(I)$. (Recall that $(\mathbf{h}_n)$ was chosen as a bounded sequence in $\mathscr{W}^p(I)$, so that (h_n) is bounded in $\widehat{\mathscr{W}^p}(I)$ and hence in $\widehat{\mathscr{X}^p}(I)$.)

Altogether, we have that for sufficiently large n and m,

$$\big\|g_n - g_m\big\|^p_{\widehat{\mathscr{X}^p}(I)} < \epsilon + \epsilon^p M^p. \tag{3.3.11}$$

Hence (g_n) is Cauchy in $\widehat{\mathscr{X}^p}(I)$, which completes the proof. $\square$

Theorem 3.3.2 (Index Zero). *Suppose that A and G satisfy the conditions (3.2.21) and (3.2.22) on page 133. If $D\Phi_{A,G}(\mathbf{0}) = \frac{\mathrm{d}}{\mathrm{d}t} - A^\sharp$ is an isomorphism of $\mathscr{W}_0^p\big([0,\infty)\big)$ onto $\mathscr{X}^p\big([0,\infty)\big)$, then $\Phi_{A,G}$ is a C^1 Fredholm map of index zero, as a map from $\mathscr{W}_0^p\big([0,\infty)\big)$ into $\mathscr{X}^p\big([0,\infty)\big)$.*

Proof. This is a direct consequence of Lemma 3.3.1 and of the invariance of the Fredholm index under compact perturbations. □

Remark 3.3.3. For a general treatment of whether $DF(\mathbf{0}) = \frac{\mathrm{d}}{\mathrm{d}t} - A^\sharp$ is an isomorphism of $\mathscr{W}_0^p\big([0,\infty)\big)$ onto $\mathscr{X}^p\big([0,\infty)\big)$, see Rabier [Rab04b], Corollary 8.5. A helpful discussion of how to satisfy the conditions in that theorem can be found in the final section of Rabier [Rab03]. In particular, it is noted there that the Laplacian $A(t) = \Delta$ satisfies most of these conditions,[3] particularly the condition involving Rademacher boundedness. Since also the spectrum of $\Delta\colon W^p \to X^p$ lies on the negative real axis[4], Corollary 8.5 of Rabier [Rab04b] asserts that $\frac{\mathrm{d}}{\mathrm{d}t} - \Delta$ is an isomorphism of $\mathscr{W}_0^p\big([0,\infty)\big)$ onto $\mathscr{X}^p\big([0,\infty)\big)$. ◊

3.3.1 Nonzero initial values

The machinery that we have built to this point could be used to prove the existence of solutions to the problem (3.0.1) with initial value $g = 0$. Our goal is to prove the existence of solutions to an initial value problem with more general initial conditions. For this reason, we now study the Fredholm properties of the augmented operator $\big(\Phi_{A,G}, E_0\big)$. The key is to identify the correct target space for the map

$$E_0(\mathbf{u}) = \mathbf{u}(0), \quad \mathbf{u} \in \mathscr{W}^p\big([0,\infty)\big). \tag{3.3.12}$$

[3] Not all of the hypotheses are mentioned in [Rab03] because of the difference in setting

[4] See Evans [Eva98], Section 6.5.1 for $p = 2$; by regularity the spectrum is independent of $p \in (1,\infty)$.

As we have already seen, because of the continuity of functions in $W^{1,p}\big([0, \infty), L^p(\Omega)\big)$, the linear function E_0 is well-defined and continuous as a map into $L^p(\Omega)$. However, the space $L^p(\Omega)$ is too large in the sense that E_0 is far from being onto $L^p(\Omega)$; recall that the functions in $\widehat{\mathscr{W}^p}\big([0, \infty)\big)$ are continuous on $[0, \infty) \times \Omega$. On the other hand, the space $W^p(\Omega) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ is too small, in the sense that the evaluation map E_0 is not continuous as a map into W^p. For this reason, we bring in the intermediate "trace" space

$$Y^p := \operatorname{rge} E_0 \subset L^p(\Omega), \tag{3.3.13}$$

with norm

$$\|g\|_{Y^p} := \inf \left\{ \|\mathbf{u}\|_{\mathscr{W}^p\left([0,\infty)\right)} : \mathbf{u}(0) = g \right\}. \tag{3.3.14}$$

For a general discussion of trace spaces for anisotropic Sobolev spaces, see Besov, Il′in and Nikol′skiĭ [BIN78, BIN79]. Because we are interested in a specific trace space, the arguments can be simplified and are therefore presented here. Also, Rabier notes in [Rab04b] that Y^p contains the real interpolation space $(X^p, W^p)_{1-(1/p),\,p}$. See Section 1.6.2 of Triebel [Tri92] for a definition, and see also Lemma 2.1 of Di Giorgio, Lunardi, and Schnaubelt [DGLS].

Lemma 3.3.4 (The Space $\mathbf{Y^p}$). *The quantity defined in* (3.3.14) *is a norm on the subspace* $Y^p := \big\{\mathbf{u}(0) : \mathbf{u} \in \mathscr{W}^p\big([0, \infty)\big)\big\}$ *of* $L^p(\Omega)$*. Moreover,* Y^p *is a Banach space when equipped with this norm.*

Proof. Since the evaluation map $E_0(\mathbf{u}) = \mathbf{u}(0)$ is continuous from $\mathscr{W}^p\big([0, \infty)\big)$ into $X^p = L^p(\Omega)$, its null-space $\ker E_0$ is closed in $\mathscr{W}^p\big([0, \infty)\big)$. Thus, the quotient $\mathscr{W}^p\big([0, \infty)\big)/\ker E_0$ is a Banach space, when equipped with the norm

$$\|\mathbf{U}\|_{\div} = \inf_{\mathbf{u}\in\mathbf{U}} \|\mathbf{u}\|_{\mathscr{W}^p\left([0,\infty)\right)}. \tag{3.3.15}$$

For this, see Kato [Kat95], section III.1.8. Moreover, E_0 induces a canonical continuous linear map, denoted by Q_0, on $\mathscr{W}^p\big([0,\,\infty)\big)/\ker E_0$ via

$$Q_0\mathbf{U} = E_0\mathbf{u} = \mathbf{u}(0), \tag{3.3.16}$$

where $\mathbf{u}$ is any vector in the equivalence class $\mathbf{U}$. Since $\ker E_0$ has been factored out, Q_0 is a bijection onto its range Y^p. Notice that for each $g \in Y^p$,

$$\begin{aligned} \|g\|_{Y^p} &:= \inf\left\{\|\mathbf{u}\|_{\mathscr{W}^p\left([0,\,\infty)\right)} : \mathbf{u}(0) = g\right\} && (3.3.17)\\ &= \inf\left\{\|\mathbf{u}\|_{\mathscr{W}^p\left([0,\,\infty)\right)} : \mathbf{u} \in Q_0^{-1}(g)\right\} && (3.3.18)\\ &= \left\|Q_0^{-1}g\right\|_{\div}. && (3.3.19)\end{aligned}$$

Since Q_0 is a linear bijection, and since $\|\cdot\|_{\div}$ is a complete norm on the domain of Q_0, this shows that $\|\cdot\|_{Y^p}$ is a complete norm on the range Y^p of Q_0. □

Lemma 3.3.5 (Evaluation Map). *The map $E_0 : \mathscr{W}^p\big([0,\,\infty)\big) \to Y^p$ is continuous, linear, and surjective.*

Proof. The linearity is clear from the definition of E_0, and the surjectivity is clear from the definition of Y^p. Since (3.3.14) implies that

$$\|E_0\mathbf{u}\|_{Y^p} \leq \|\mathbf{u}\|_{\mathscr{W}^p\left([0,\,\infty)\right)}, \tag{3.3.20}$$

continuity holds as well. □

Since E_0 is continuous and linear, it is C^1, with derivative $DE_0(\mathbf{u})\mathbf{h} = \mathbf{h}(0)$. We thus have the following.

Lemma 3.3.6 (Augmented Operator). *The operator*

$$\big(\Phi_{A,G},\, E_0\big) : \mathscr{W}^p\big([0,\,\infty)\big) \to \mathscr{X}^p\big([0,\,\infty)\big) \times Y^p \tag{3.3.21}$$

is continuously differentiable, with derivative

$$D\big((\Phi_{A,G},\, E_0)\big) = \big(D\Phi_{A,G},\, E_0\big). \tag{3.3.22}$$

As for the Fredholm property for this operator, we first note that

$$D\big((\Phi_{A,G},\, E_0)\big)(\mathbf{u}) - D\big((\Phi_{A,G},\, E_0)\big)(\mathbf{0}) = \big(D\Phi_{A,G}(\mathbf{u}) - D\Phi_{A,G}(\mathbf{0}), 0\big). \tag{3.3.23}$$

Hence, the compactness of the linear operator in (3.3.23) follows from the compactness of $D\Phi_{A,G}(\mathbf{u}) - D\Phi_{A,G}(\mathbf{0})$, which was proved in Lemma 3.3.1.

The last thing to consider at this point is the question of whether the derivative of our augmented operator, evaluated at $\mathbf{0}$, is an isomorphism and hence Fredholm of index zero. Lemma 3.3.7 will show that this depends only on the answer to the same question for $\Phi_{A,G}$, though with respect to different domains and ranges:

Lemma 3.3.7 (Augmented Linear Isomorphism). *If $\frac{\mathrm{d}}{\mathrm{dt}} - A^\sharp$ is an isomorphism of $\mathscr{W}_0^p\big([0,\,\infty)\big)$ onto $\mathscr{X}^p\big([0,\,\infty)\big)$, then $D\big(\frac{\mathrm{d}}{\mathrm{dt}} - A^\sharp,\, E_0\big)(\mathbf{0})$ is an isomorphism of $\mathscr{W}^p\big([0,\,\infty)\big)$ onto $\mathscr{X}^p\big([0,\,\infty)\big) \times Y^p$.*

Proof. That $\frac{\mathrm{d}}{\mathrm{dt}} - A^\sharp$ is an isomorphism of $\mathscr{W}_0^p\big([0,\,\infty)\big)$ onto $\mathscr{X}^p\big([0,\,\infty)\big)$ implies that for each function $\mathbf{f} \in \mathscr{X}^p\big([0,\,\infty)\big)$, there is a unique $\mathbf{v} \in \mathscr{W}_0^p\big([0,\,\infty)\big)$ such that

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathbf{v} - A^\sharp\mathbf{v} = \mathbf{f}. \tag{3.3.24}$$

Now let $(\mathbf{f},\, g) \in \mathscr{X}^p\big([0,\,\infty)\big) \times Y^p$ be given. By the definition (3.3.13) of Y^p, let $\mathbf{h} \in \mathscr{W}^p\big([0,\,\infty)\big)$ be such that

$$\mathbf{h}(0) = g. \tag{3.3.25}$$

Since $\frac{\mathrm{d}}{\mathrm{dt}}\mathbf{h}$ and $A^\sharp\mathbf{h}$ are well-defined elements of $\mathscr{X}^p\big([0,\,\infty)\big)$, it follows by assumption that there is a unique $\mathbf{v} \in \mathscr{W}_0^p\big([0,\,\infty)\big)$ such that

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathbf{v} - A^\sharp\mathbf{v} = \mathbf{f} - \frac{\mathrm{d}}{\mathrm{dt}}\mathbf{h} + A^\sharp\mathbf{h}. \tag{3.3.26}$$

We set $\mathbf{u} = \mathbf{v} + \mathbf{h}$; it follows at once that $\mathbf{u}(0) = g$ and

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathbf{u} - A^\sharp\mathbf{u} = \mathbf{f}. \tag{3.3.27}$$

Finally, this solution $\mathbf{u}$ is unique, since the difference $\mathbf{u}_1 - \mathbf{u}_2$ of two solutions gives a solution in $\mathscr{W}_0^p\big([0,\,\infty)\big)$ to

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathbf{v} - A^\sharp\mathbf{v} = \mathbf{0}, \tag{3.3.28}$$

which, by assumption, has only the trivial solution. $\square$

Altogether, we have the following result to complement Theorem 3.3.2:

Theorem 3.3.8 (Index Zero). *Suppose that A and G satisfy the conditions (3.2.21) and (3.2.22) on page 133. If $D\Phi_{A,G}(\mathbf{0}) = \frac{d}{dt} - A^\sharp$ is an isomorphism of the space $\mathscr{W}_0^p\big([0,\,\infty)\big)$ onto the space $\mathscr{X}^p\big([0,\,\infty)\big)$, then $\big(\Phi_{A,G},\, E_0\big)$ is a C^1 Fredholm map of index zero, as a map from $\mathscr{W}^p\big([0,\,\infty)\big)$ into $\mathscr{X}^p\big([0,\,\infty)\big) \times Y^p$.*

3.4 PROPERNESS ON THE CLOSED BOUNDED SUBSETS

To continue our preparation of an application of the degree theory, we must know when $\left(\Phi_{A,G},\, E_0\right)$ is proper on the closed bounded subsets of $\mathscr{W}^p\big([0,\, \infty)\big)$. We begin with the following lemma, taken from from Rabier [Rab04a]. In particular, see the remarks at the end of Section 4 of that paper, where Rabier explains how the following is obtained as a generalization of his Theorem 9.

Lemma 3.4.1. *Suppose that E is a reflexive Banach space, that $p \in (1,\, \infty)$, and that S is any given δ-net. Suppose that $\mathcal{H}$ is a bounded subset of $W^{1,p}(\mathbb{R},\, E)$ and that $\mathcal{H}(\mathbb{R})$ is relatively compact in E. The following conditions are equivalent:*

1. *$\mathcal{H}$ is relatively compact subset of $C_{\{0\}}(\mathbb{R},\, E)$.*
2. *For any $\mathbf{u} \in W^{1,p}(\mathbb{R},\, E)$, if there exists a sequence $(\mathbf{u}_n) \subset \mathcal{H}$ and a sequence $(\xi_n) \subset S$ such that $\lim_{n\to\infty} \left|\xi_n\right| = \infty$ and such that*

$$\tau_{\xi_n}\mathbf{u}_n \overset{w}{\rightharpoonup} \mathbf{u} \text{ in } W^{1,p}(\mathbb{R},\, E) \text{ as } n \to \infty \tag{3.4.1}$$

then $\mathbf{u} = \mathbf{0}$.

Remark 3.4.2. In applications of Lemma 3.4.1, $\mathcal{H}$ is often arranged as a sequence $(\mathbf{h}_n)$. If so, one need only consider those sequences $(\mathbf{u}_n)$ drawn from $\mathcal{H}$ that are also subsequences of $(\mathbf{h}_n)$. To see this, note that an arbitrary subsequence $(\mathbf{u}_n)$ drawn from $\mathcal{H}$ either does or does not possess a subsequence that is a subsequence of $(\mathbf{h}_n)$. If $(\mathbf{u}_n)$ does not possess such a subsequence, then $(\mathbf{u}_n)$ possesses a constant subsequence $\mathbf{u}_{n_k} \equiv \mathbf{v}$. Because $\mathbf{v}(t) \to 0$ as $\left|t\right| \to \infty$, it follows that

$$\tau_{\xi_n}\mathbf{u}_n \overset{w}{\rightharpoonup} \mathbf{0} \text{ in } W^{1,p}(\mathbb{R},\, E) \text{ as } n \to \infty, \tag{3.4.2}$$

in which case the desired conclusion follows by the existence of weak limits. Thus, we are left only with the possibility that

$$\tau_{\xi_k}\mathbf{u}_{n_k} \xrightarrow{w} \mathbf{u} \text{ in } W^{1,p}(\mathbb{R}, E) \text{ as } k \to \infty, \tag{3.4.3}$$

where $(\mathbf{u}_{n_k})$ is a subsequence of $(\mathbf{h}_n)$. That it suffices to only consider this case is the content of this remark. $\Diamond$

Let $\mathbf{v} \in L^p\big([0, \infty), E\big)$, and define

$$\mathcal{E}\mathbf{v}(t) := \begin{cases} \mathbf{v}(t), \quad t \geq 0, \\ (t+1)\mathbf{v}(-t), \quad -1 \leq t < 0, \\ 0, \quad t < -1. \end{cases} \tag{3.4.4}$$

The operator $\mathcal{E}$ evidently has the following properties:

Lemma 3.4.3. *The operator $\mathcal{E}$ is a bounded linear extension operator from $L^p\big([0, \infty), E\big)$ into $L^p(\mathbb{R}, E)$. The image of $L^p\big([0, \infty), E\big)$ under $\mathcal{E}$ consists entirely of functions that vanish almost everywhere on the interval $(-\infty, -1)$. Moreover, $\mathcal{E}$ may also be viewed as a bounded linear extension operator from $W^{1,p}\big([0, \infty), E\big)$ into $W^{1,p}(\mathbb{R}, E)$, or from $C_b\big([0, \infty), E\big)$ into $C_b(\mathbb{R}, E)$.*

Now let E' be a subspace of E, with a norm (possibly not the norm induced by that of E) that makes E' into a reflexive Banach space. Since the above lemma applies with E replaced by E', we have the following corollary, used implicitly during the proof of Theorem 3.4.9.

Corollary 3.4.4. *The operator $\mathcal{E}$ is a bounded linear extension operator from*

$$W^{1,p}\big([0, \infty), E\big) \cap L^p\big([0, \infty), E'\big)$$

into the subspace of $W^{1,p}(\mathbb{R}, E) \cap L^p(\mathbb{R}, E')$ consisting of functions vanishing on $(-\infty, -1)$.

With use of this extension operator, we can prove the following as a result of Lemma 3.4.1.

Lemma 3.4.5. *Let $\mathcal{H}$ be a bounded subset of $W^{1,p}\big([0,\infty),\, E\big)$, where E is a reflexive Banach space and $1 < p < \infty$. Suppose that $\mathcal{H}\big([0,\infty)\big)$ is relatively compact in E. Then the following are equivalent:*

1. *The set $\mathcal{H}$ is a relatively compact subset of $C_{\{0\}}\big([0,\infty),\, E\big)$.*
2. *If $\mathbf{u} \in W^{1,p}(\mathbb{R},\, E)$ is such that there exist a sequence $(\mathbf{u}_n) \subset \mathcal{H}$ and a sequence $(\xi_n) \subset [0,\infty)$ with $\xi_n \to \infty$ where*

$$\tau_{\xi_n}\mathcal{E}\mathbf{u}_n \overset{w}{\rightharpoonup} \mathbf{u} \text{ in } W^{1,p}(\mathbb{R},\, E) \text{ as } n \to \infty, \tag{3.4.5}$$

then $\mathbf{u} = \mathbf{0}$.

Proof. For $2 \Rightarrow 1$, we first use Lemma 3.4.1 to show that the subset $\mathcal{E}(\mathcal{H})$ of $W^{1,p}(\mathbb{R},\, E)$ is relatively compact in $C_{\{0\}}(\mathbb{R},\, E)$. Certainly, $\mathcal{E}(\mathcal{H})$ is bounded, since $\mathcal{H} \subset W^{1,p}\big([0,\infty),\, E\big)$ is bounded and $\mathcal{E} \in \mathcal{L}\big(W^{1,p}\big([0,\infty),\, E\big),\, W^{1,p}(\mathbb{R},\, E)\big)$. Next, to see that $\mathcal{E}\mathcal{H}(\mathbb{R})$ is relatively compact in E, we use the definition of $\mathcal{E}$ to see that

$$\mathcal{E}\mathcal{H}(\mathbb{R}) = \mathcal{H}\big([0,\infty)\big) \cup \{(t+1)\mathbf{v}(-t) \colon\ -1 \le t \le 0,\ \mathbf{v} \in \mathcal{H}\} \tag{3.4.6}$$

$$\subset [0,\,1]\mathcal{H}\big([0,\infty)\big). \tag{3.4.7}$$

Since $\mathcal{H}\big([0,\infty)\big)$ is assumed relatively compact in E, and $[0,\,1]$ is compact in $\mathbb{R}$, and scalar multiplication is continuous from $\mathbb{R} \times E$ into E, this shows that $\mathcal{E}\mathcal{H}(\mathbb{R})$ is relatively compact in E.

To complete the application of Lemma 3.4.1, suppose that $\mathbf{u} \in W^{1,p}(\mathbb{R}, E)$ is such that there exists a sequence $(\mathcal{E}\mathbf{u}_n) \subset \mathcal{E}(\mathcal{H})$ and a sequence $(\xi_n) \subset \mathbb{R}$ such that $|\xi_n| \to \infty$ and that

$$\tau_{\xi_n}\mathcal{E}\mathbf{u}_n \xrightarrow{w} \mathbf{u} \text{ in } W^{1,p}(\mathbb{R}, E) \text{ as } n \to \infty. \tag{3.4.8}$$

We are to show that $\mathbf{u} = \mathbf{0}$. If there is a subsequence (ξ_{n_k}) such that $\xi_{n_k} \to \infty$, then $\mathbf{u} = \mathbf{0}$ follows directly from the available assumption (2) of the present lemma. Otherwise, it must be that $\xi_n \to -\infty$. Recall from the definition of $\mathcal{E}$ that all of the functions $\mathcal{E}\mathbf{u}_n$ are supported in $(-1, \infty)$. Hence, the sequence of translates $\tau_{\xi_n}\mathcal{E}\mathbf{u}_n$ converges to zero uniformly on compact sets, whence $\mathbf{u}$ again equals $\mathbf{0}$.

We conclude from Lemma 3.4.1 that $\mathcal{E}(\mathcal{H})$ is relatively compact in $C_{\{0\}}(\mathbb{R}, E)$ and that therefore for any sequence $(\mathbf{v}_n) \subset \mathcal{H}$ there is a subsequence $(\mathbf{v}_{n_k})$ such that the sequence $(\mathcal{E}\mathbf{v}_{n_k})$ of extensions to $\mathbb{R}$ is convergent in $C_{\{0\}}(\mathbb{R}, E)$. Hence, by restricting back to $[0, \infty)$, the sequence $(\mathbf{v}_{n_k})$ is convergent in $C_{\{0\}}\big([0, \infty), E\big)$. This establishes the relative compactness of $\mathcal{H}$ in $C_{\{0\}}\big([0, \infty), E\big)$.

For the implication 1 $\Rightarrow$ 2, notice first that the continuity of the extension operator $\mathcal{E}$ proves that it preserves relative compactness. Hence, $\mathcal{E}(\mathcal{H})$ is relatively compact in $C_{\{0\}}(\mathbb{R}, E)$. The equivalent conditions of Lemma 3.4.1 therefore hold. Evidently, Condition 2 of Lemma 3.4.1 implies Condition 2 by restriction to sequences tending to $+\infty$. $\square$

We shall need the following commutativity properties.

Lemma 3.4.6. *Let* $\mathbf{u} \in \mathscr{W}^p\big([0, \infty)\big)$, $\mathbf{v} \in \mathscr{W}^p(\mathbb{R})$, *and* $\xi \in \mathbb{R}$. *Then*

$$\frac{d}{dt}\tau_\xi \mathbf{v} = \tau_\xi \frac{d}{dt}\mathbf{v} \text{ in } \mathscr{X}^p(\mathbb{R}), \tag{3.4.9}$$

and

$$\left(\mathcal{E}\,\frac{d}{\mathrm{dt}}\,\mathbf{u}\right)(t)=\left(\frac{d}{\mathrm{dt}}\,\mathcal{E}\mathbf{u}\right)(t)\ \textit{ in } L^p(\Omega),\ \textit{for almost every } t>0. \tag{3.4.10}$$

Proof. For the first assertion, let a test function $\phi \in C_0^\infty(\mathbb{R})$ be given. Then we have the following equality of Bochner integrals:

$$\begin{aligned}
\int_{\mathbb{R}} \tau_\xi \mathbf{v}(t)\phi'(t)\,\mathrm{d}t &= \int_{\mathbb{R}} \mathbf{v}(t+\xi)\phi'(t)\,\mathrm{d}t && (3.4.11)\\
&= \int_{\mathbb{R}} \mathbf{v}(s)\phi'(s-\xi)\,\mathrm{d}s && (3.4.12)\\
&= -\int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{dt}}\,\mathbf{v}(s)\phi(s-\xi)\,\mathrm{d}s && (3.4.13)\\
&= -\int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{dt}}\,\mathbf{v}(t+\xi)\phi(t)\,\mathrm{d}t && (3.4.14)\\
&= -\int_{\mathbb{R}} \tau_\xi\,\frac{\mathrm{d}}{\mathrm{dt}}\,\mathbf{v}(t)\phi(t)\,\mathrm{d}t. && (3.4.15)
\end{aligned}$$

This proves that $\frac{\mathrm{d}}{\mathrm{dt}}\,\tau_\xi\mathbf{v}=\tau_\xi\,\frac{\mathrm{d}}{\mathrm{dt}}\,\mathbf{v}$, by definition. For the second assertion, let a test function $\psi \in C_0^\infty(\Omega)$ be given. Let $t>0$. Then

$$\begin{aligned}
\int_{\Omega} (\mathcal{E}\mathbf{u})(t)\psi'(t)\,\mathrm{d}t &= \int_{\Omega} \mathbf{u}(t)\psi'(t)\,\mathrm{d}t && (3.4.16)\\
&= -\int_{\Omega} \left(\frac{\mathrm{d}}{\mathrm{dt}}\,\mathbf{u}\right)(t)\psi(t)\,\mathrm{d}t && (3.4.17)\\
&= -\int_{\Omega} \left(\mathcal{E}\,\frac{\mathrm{d}}{\mathrm{dt}}\,\mathbf{u}\right)(t)\psi(t)\,\mathrm{d}t. && (3.4.18)
\end{aligned}$$

□

We will make use of one more lemma to prove Theorem 3.4.9. The lemma is used in the third step of the proof of that theorem.

Lemma 3.4.7. *Let $H : [0, \infty) \to \mathcal{L}\big(L^p(\Omega)\big)$ be bounded and such that*

$$\lim_{t\to\infty} \big\|H(t)\big\|_{op} = 0. \tag{3.4.19}$$

Then pointwise multiplication by H defines a compact linear operator from $\mathscr{W}^p\big([0, \infty)\big)$ into $\mathscr{X}^p\big([0, \infty)\big)$.

Proof. Let $(\mathbf{u}_n)$ be a sequence drawn from the unit ball of $\mathscr{W}^p\big([0, \infty)\big)$. We are to prove that for $\mathbf{v}_n(t) := A(t)\mathbf{u}_n(t)$, the sequence $(\mathbf{v}_n)$ has an $\mathscr{X}^p\big([0, \infty)\big)$-Cauchy subsequence.

We begin with a diagonal argument. According to Simon ([Sim87], Theorem 1), the space $\mathscr{W}^p([0, 1])$ is compactly contained in $\mathscr{X}^p([0, 1])$. Hence, there is a subsequence $(\mathbf{u}_{1;n})$ that converges in $\mathscr{X}^p([0, 1])$. Of course, it is actually the sequence of restrictions that is convergent, but this is what we shall mean when we say that a sequence from $\mathscr{X}^p\big([0, \infty)\big)$ converges in $\mathscr{X}^p([0, T])$.

Inductively, having defined the sequence $(\mathbf{u}_{m;n})$, we use the compact containment of $\mathscr{W}^p([0, m + 1])$ in $\mathscr{X}^p([0, m + 1])$ to extract a subsequence $(\mathbf{u}_{m+1;n})$ that converges in $\mathscr{X}^p([0, m + 1])$.

From the construction, the following facts about the sequences $(\mathbf{u}_{m;n})$ are clear. First, if $m_1 < m_2$, then $(\mathbf{u}_{m_2;n})$ is a subsequence of $(\mathbf{u}_{m_1;n})$. Also, and as a result, if $m_1 < m_2$ then $(\mathbf{u}_{m_2;n})$ converges in $\mathscr{X}^p([0, m_1])$. From these it follows that the diagonal sequence $(\mathbf{w}_n) := (\mathbf{u}_{n;n})$ is eventually a subsequence of each of the sequences $(\mathbf{u}_{m;n})$, and is hence convergent in each of the spaces $\mathscr{X}^p([0, m])$.

We claim that $(\mathbf{y}_n) := (\mathbf{v}_{n;n})$ is the desired $\mathscr{X}^p\big([0, \infty)\big)$-Cauchy subsequence of $(\mathbf{v}_n)$. Let $\epsilon > 0$. By assumption on H, there is some $N = N(\epsilon) \in \mathbb{N}$ such that $\big\|H(t)\big\|_{\mathcal{L}\left(L^p(\Omega)\right)} < \epsilon/4$

whenever $t > N$. Since $(\mathbf{w}_n)$ is convergent in $\mathscr{X}^p([0, N])$, we can find N_1 such that for all $n, m > N_1$,

$$\left\|\mathbf{w}_n - \mathbf{w}_m\right\|_{\mathscr{X}^p([0, N])} < \epsilon/2M, \tag{3.4.20}$$

where M is a bound for $\left\|H(t)\right\|_{\mathcal{L}\left(L^p(\Omega)\right)}$, which is assumed to exist. Let $n, m > N_1$. To estimate

$$\left\|\mathbf{y}_n - \mathbf{y}_m\right\|^p_{\mathscr{X}^p\left([0, \infty)\right)} = \int_0^\infty \left\|H(t)(\mathbf{w}_n(t) - \mathbf{w}_m(t))\right\|^p_{L^p(\Omega)} \mathrm{d}t, \tag{3.4.21}$$

we integrate first on $[0, N]$ and then on $[N, \infty)$. First,

$$\begin{aligned}
&\int_0^N \| \; H(t)(\mathbf{w}_n(t) \; -\mathbf{w}_m(t)) \; \|^p_{L^p(\Omega)} \mathrm{d}t \\
&\qquad \le \int_0^N \left\|H(t)\right\|^p_{\mathcal{L}\left(L^p(\Omega)\right)} \left\|\mathbf{w}_n(t) - \mathbf{w}_m(t)\right\|^p_{L^p(\Omega)} \mathrm{d}t && (3.4.22) \\
&\qquad \le M^p \left\|\mathbf{w}_n - \mathbf{w}_m\right\|^p_{\mathscr{X}^p([0, N])} && (3.4.23) \\
&\qquad \le M^p(\epsilon/2M)^p = (\epsilon/2)^p. && (3.4.24)
\end{aligned}$$

Second,

$$\begin{aligned}
&\int_N^\infty \| \; H(t)(\mathbf{w}_n(t) \; -\mathbf{w}_m(t)) \; \|^p_{L^p(\Omega)} \mathrm{d}t \\
&\qquad \le \int_N^\infty \left\|H(t)\right\|^p_{\mathcal{L}\left(L^p(\Omega)\right)} \left\|\mathbf{w}_n(t) - \mathbf{w}_m(t)\right\|^p_{L^p(\Omega)} \mathrm{d}t && (3.4.25) \\
&\qquad \le (\epsilon/4)^p \left\|\mathbf{w}_n - \mathbf{w}_m\right\|^p_{\mathscr{X}^p([N, \infty])} && (3.4.26) \\
&\qquad \le (\epsilon/4)^p \left\|\mathbf{w}_n - \mathbf{w}_m\right\|^p_{\mathscr{X}^p\left([0, \infty)\right)} && (3.4.27) \\
&\qquad \le (\epsilon/2)^p, && (3.4.28)
\end{aligned}$$

since $(\mathbf{w}_n)$ is drawn from the unit ball of $\mathscr{W}^p(\mathbb{R})$, which is contained in the unit ball of $\mathscr{X}^p\left([0, \infty)\right)$. Altogether,

$$\left\|\mathbf{y}_n - \mathbf{y}_m\right\|_{\mathscr{X}^p\left([0, \infty)\right)} \le ((\epsilon/2)^p + (\epsilon/2)^p)^{1/p} < \epsilon, \tag{3.4.29}$$

as desired. $\square$

We are now in a position to prove the required properness result for $\Phi_{A,G}$. Let us recall the definition (3.2.23) of the function $\Phi_{A,G}$. We also recall the set $\omega(G)$, as defined in Section 2.2.1 on page 26. We redefine the phrase "admissible omega-limit set", originally introduced in Section 2.2 for a similar purpose, as follows:

Definition 3.4.8. Suppose that for all compact subsets K of $\mathbb{R}$, the function $G : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ is bounded and uniformly continuous on $[0, \infty) \times K$. We say that G has an *admissible* omega-set $\omega(G)$ (relative to A) if the following condition is satisfied:

If $\mathbf{u} \in \mathscr{W}^p(\mathbb{R})$ is such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} - A^{\infty\sharp}\mathbf{u} + G^{\infty\sharp}(\mathbf{u}) = \mathbf{0} \tag{3.4.30}$$

for some $G^\infty \in \omega(G)$, then $\mathbf{u} = \mathbf{0}$. ♦

Theorem 3.4.9. *Assume that the three conditions (3.2.21) hold of A and that the three conditions (3.2.22) hold of G. We assume moreover that the linear operator $\left(\frac{d}{\mathrm{d}t} - A^\sharp\right)$ is an isomorphism of $\mathscr{W}_0^p\big([0, \infty)\big)$ onto $\mathscr{X}^p\big([0, \infty)\big)$. Also assume that G has an admissible omega-limit set $\omega(G)$, as defined just above.*

Then $(\Phi_{A,G}, E_0)$ is proper on the closed bounded subsets of $\mathscr{W}^p\big([0, \infty)\big)$.

Proof. Let $(\mathbf{u}_n)$ be a bounded sequence in $\mathscr{W}^p\big([0, \infty)\big)$, and put

$$\mathbf{f}_n := \Phi_{A,G}(\mathbf{u}_n) \tag{3.4.31}$$

$$= \left(\frac{\mathrm{d}}{\mathrm{d}t} - A^\sharp\right)\mathbf{u}_n + G^\sharp(\mathbf{u}_n). \tag{3.4.32}$$

Supposing that $(\mathbf{f}_n,\, \mathbf{u}_n(0))$ converges in $\mathscr{X}^p\big([0,\,\infty)\big)\times Y^p$ to some $(\mathbf{f},\, g)\in\mathscr{X}^p\big([0,\,\infty)\big)\times Y^p$, we seek a subsequence of $(\mathbf{u}_n)$ that converges in $\mathscr{W}^p\big([0,\,\infty)\big)$. This will establish the properness of $\big(\Phi_{A,G},\, E_0\big)$ on the closed bounded subsets of $\mathscr{W}^p\big([0,\,\infty)\big)$. (See Proposition 1.4.8 on page 11.) We proceed in several steps.

Step 1. We first show that there is a subsequence of $(\mathbf{u}_n)$ that converges in the space $C_{\{0\}}\big([0,\,\infty), L^p(\Omega)\big)$. We will do this by application of Lemma 3.4.5 to the set $\mathcal{H}$ that consists of the vectors in the sequence $(\mathbf{u}_n)$, which we have already assumed is bounded. To see that $\mathcal{H}\big([0,\,\infty)\big)$ is relatively compact in X^p, suppose instead that there is a subsequence $(\mathbf{u}_{n_k})$ and a sequence $(\xi_k)\subset[0,\,\infty)$ such that the sequence $(\mathbf{u}_{n_k}(\xi_k))\subset X^p$ has no convergent subsequence. Consider then the following bounded sequence in $\mathscr{W}^p(\mathbb{R})$:

$$\mathbf{v}_k := \tau_{\xi_k}\mathcal{E}\mathbf{u}_{n_k}. \tag{3.4.33}$$

By restriction, consider $(\mathbf{v}_k)$ as a bounded sequence in $\mathscr{W}^p(I)$, where $I=(-1,\,1)$. According to Simon [Sim87], the space $\mathscr{W}^p(I)$ is compactly contained in $C_b(I,\, X^p)$ because I is bounded. In particular, $(\mathbf{v}_k(0))$ has a subsequence convergent in X^p. This is impossible, because

$$\mathbf{v}_k(0) = \tau_{\xi_k}\mathcal{E}\mathbf{u}_{n_k}(0) = \mathcal{E}\mathbf{u}_{n_k}(\xi_k) = \mathbf{u}_{n_k}(\xi_k), \tag{3.4.34}$$

which was assumed to have no convergent subsequence.

To complete our application of Lemma 3.4.5, suppose that we have $\mathbf{u}\in W^{1,p}(\mathbb{R},\, X^p)$, $(\mathbf{u}_{n_k})\subset\mathcal{H}$, and $(\xi_k)\subset[0,\,\infty)$ such that $\lim_{k\to\infty}\xi_k=\infty$, and such that

$$\mathbf{v}_k := \tau_{\xi_k}\mathcal{E}\mathbf{u}_{n_k} \xrightarrow{w} \mathbf{u} \text{ in } W^{1,p}(\mathbb{R},\, X^p) \text{ as } k\to\infty. \tag{3.4.35}$$

Since $(\mathbf{v}_k)$ is bounded in $\mathscr{W}^p(\mathbb{R})$, it is no loss of generality to assume that $(\mathbf{v}_k)$ converges weakly to $\mathbf{u}$ in $\mathscr{W}^p(\mathbb{R})$, not just in $W^{1,p}(\mathbb{R},\, X^p)$.

According to Lemma 2.2.5 on page 27, it is also no loss of generality to assume that $\tau_{\xi_k}\mathcal{E}G$ converges uniformly on compact sets to some $G^\infty \in \omega(G)$, which we now take to be fixed. Since G^∞ is defined on all of $\mathbb{R}\times\mathbb{R}$, and satisfies all of the hypotheses of Lemma 3.2.3, the Nemytskii operator $G^{\infty\sharp}$ is well defined on $\mathscr{W}^p(\mathbb{R})$. Put

$$\mathbf{h} := \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} - A^{\infty\sharp}\mathbf{u} + G^{\infty\sharp}(\mathbf{u}) \in \mathscr{X}^p(\mathbb{R}), \tag{3.4.36}$$

and

$$\begin{aligned}\mathbf{h}_k &:= \tau_{\xi_k}\mathcal{E}\mathbf{f}_{n_k} && (3.4.37)\\ &= \tau_{\xi_k}\mathcal{E}\big((\frac{\mathrm{d}}{\mathrm{dt}} - A^\sharp)\mathbf{u}_{n_k} + G^\sharp(\mathbf{u}_{n_k})\big) \in \mathscr{X}^p(\mathbb{R}). && (3.4.38)\end{aligned}$$

We are going to prove that $\mathbf{h}$ is the weak limit in $\mathscr{X}^p(\mathbb{R})$ of the sequence $(\mathbf{h}_k)$. To see this, we consider in turn each of three terms that sum to $\mathbf{h}_k$. We first show that

$$\tau_{\xi_k}\mathcal{E}\,\frac{\mathrm{d}}{\mathrm{dt}}\,\mathbf{u}_{n_k} \xrightarrow{w} \frac{\mathrm{d}}{\mathrm{dt}}\,\mathbf{u} \text{ in } \mathscr{X}^p(\mathbb{R}) \text{ as } k\to\infty. \tag{3.4.39}$$

Recall that (see Edwards [Edw65], Section 8.20) since $\mathscr{X}^p(\mathbb{R}) = L^p(\mathbb{R}, X^p)$ and since X^p is reflexive, $\mathscr{X}^p(\mathbb{R})^*$ may be represented by $L^{p'}(\mathbb{R},\, X^{p'})$, which has $C_0^\infty(\mathbb{R},\, X^{p'})$ as a dense subspace. Let $\boldsymbol{\phi} \in C_0^\infty(\mathbb{R},\, X^{p'})$, and let $T > 0$ be such that $\operatorname{supp}\boldsymbol{\phi} \subset [-T,\, T]$. Let k be sufficiently large that $-T+\xi_k>0$. Then the action of $\boldsymbol{\phi}$ on $\tau_{\xi_k}\mathcal{E}\,\frac{\mathrm{d}}{\mathrm{dt}}\,\mathbf{u}_{n_k}$ is as follows:

$$\Big\langle \boldsymbol{\phi},\ \tau_{\xi_k}\mathcal{E}\,\frac{\mathrm{d}}{\mathrm{dt}}\,\mathbf{u}_{n_k}\Big\rangle = \int_{\mathbb{R}}\Big\langle \boldsymbol{\phi}(t),\ \big(\tau_{\xi_k}\mathcal{E}\,\frac{\mathrm{d}}{\mathrm{dt}}\,\mathbf{u}_{n_k}\big)\Big\rangle(t)\,\mathrm{d}t \tag{3.4.40}$$

$$= \int_{-T}^{T}\Big\langle \boldsymbol{\phi}(t),\ \big(\mathcal{E}\,\frac{\mathrm{d}}{\mathrm{dt}}\,\mathbf{u}_{n_k}\big)(t+\xi_k)\Big\rangle\,\mathrm{d}t \tag{3.4.41}$$

$$= \int_{-T}^{T}\Big\langle \boldsymbol{\phi}(t),\ \big(\frac{\mathrm{d}}{\mathrm{dt}}\,\mathcal{E}\mathbf{u}_{n_k}\big)(t+\xi_k)\Big\rangle\,\mathrm{d}t \tag{3.4.42}$$

$$= \int_{-T}^{T}\Big\langle \boldsymbol{\phi}(t),\ \big(\tau_{\xi_k}\,\frac{\mathrm{d}}{\mathrm{dt}}\,\mathcal{E}\mathbf{u}_{n_k}\big)(t)\Big\rangle\,\mathrm{d}t \tag{3.4.43}$$

$$= \int_{-T}^{T}\Big\langle \boldsymbol{\phi}(t),\ \big(\frac{\mathrm{d}}{\mathrm{dt}}\,\tau_{\xi_k}\mathcal{E}\mathbf{u}_{n_k}\big)(t)\Big\rangle\,\mathrm{d}t \tag{3.4.44}$$

$$= \Big\langle \boldsymbol{\phi},\ \frac{\mathrm{d}}{\mathrm{dt}}\,\tau_{\xi_k}\mathcal{E}\mathbf{u}_{n_k}\Big\rangle, \tag{3.4.45}$$

where we have repeatedly used commutativity properties as expressed in Lemma 3.4.6. The final expression is just $\big\langle \boldsymbol{\phi},\ \frac{\mathrm{d}}{\mathrm{dt}}\,\mathbf{v}_k\big\rangle$. Since $\frac{\mathrm{d}}{\mathrm{dt}}$ is a bounded linear operator from $\mathscr{W}^p(\mathbb{R})$ into $\mathscr{X}^p(\mathbb{R})$, and since $(\mathbf{v}_k)$ is assumed to converge weakly to $\mathbf{u}$ in $\mathscr{W}^p(\mathbb{R})$, it follows that the sequence $(\big\langle \boldsymbol{\phi},\ \frac{\mathrm{d}}{\mathrm{dt}}\,\mathbf{v}_k\big\rangle)$ converges to $\big\langle \boldsymbol{\phi},\ \frac{\mathrm{d}}{\mathrm{dt}}\,\mathbf{u}\big\rangle$. Altogether, this proves (3.4.39). Next, we show that

$$\tau_{\xi_k}\mathcal{E}A^{\sharp}\mathbf{u}_{n_k} \xrightarrow{w} A^{\infty\sharp}\mathbf{u} \text{ in } \mathscr{X}^p(\mathbb{R}) \text{ as } k\to\infty. \tag{3.4.46}$$

Once again, let $\boldsymbol{\phi}\in C_0^\infty(\mathbb{R},\,X^{p'})$, let $T>0$ be such that $\operatorname{supp}\boldsymbol{\phi}\subset[-T,\,T]$, and let k be sufficiently large that $-T+\xi_k>0$. The action of $\boldsymbol{\phi}$ on $\tau_{\xi_k}\mathcal{E}A^{\sharp}\mathbf{u}_{n_k}-A^{\infty\sharp}\mathbf{u}$ is given by

$$\int_{-T}^{T}\big\langle\big(\tau_{\xi_k}\mathcal{E}A^{\sharp}\mathbf{u}_{n_k}-A^{\infty\sharp}\mathbf{u}\big)(t),\ \boldsymbol{\phi}(t)\big\rangle\,\mathrm{d}t. \tag{3.4.47}$$

As before, we may disregard the extension operator $\mathcal{E}$ because $-T+\xi>0$. Hence, and by adding and subtracting $A^{\infty}\mathbf{u}_{n_k}(t+\xi_k)$ under the integral sign, the integral (3.4.47) may be written as

$$\int_{-T}^{T} \left\langle \big(A(t+\xi_k) - A^\infty\big)\mathbf{u}_{n_k}(t+\xi_k),\, \boldsymbol{\phi}(t) \right\rangle \mathrm{d}t + \int_{-T}^{T} \left\langle A^\infty \big(\mathbf{u}_{n_k}(t+\xi_k) - \mathbf{u}(t)\big),\, \boldsymbol{\phi}(t) \right\rangle \mathrm{d}t$$
$$= \int_{-T}^{T} \left\langle \big(A(t+\xi_k) - A^\infty\big)\mathbf{v}_k(t),\, \boldsymbol{\phi}(t) \right\rangle \mathrm{d}t + \int_{-T}^{T} \left\langle A^\infty \big(\mathbf{v}_k(t) - \mathbf{u}(t)\big),\, \boldsymbol{\phi}(t) \right\rangle \mathrm{d}t. \quad (3.4.48)$$

The first term on the right hand side of (3.4.48) is estimated as follows:

$$\int_{-T}^{T} \Big\langle \big(A(t+\xi_k) - A^\infty\big)\mathbf{v}_k(t),\, \boldsymbol{\phi}(t) \Big\rangle \,\mathrm{d}t$$
$$\leq \int_{-T}^{T} \left\| \big(A(t+\xi_k) - A^\infty\big)\mathbf{v}_k(t) \right\|_{X^p} \left\| \boldsymbol{\phi}(t) \right\|_{X^{p'}} \mathrm{d}t \quad (3.4.49)$$
$$\leq \sup_{t>-T} \left\| A(t+\xi_k) - A^\infty \right\|_{\mathcal{L}(W^p,\, X^p)} \left\| \mathbf{v}_{n_k} \right\|_{\mathscr{W}^p(\mathbb{R})} \left\| \boldsymbol{\phi} \right\|_{\mathscr{X}^p(\mathbb{R})^*}. \quad (3.4.50)$$

Here, the first factor tends to zero as $k \to \infty$ because of the convergence of $A(t)$ to A^∞. Recall that the second factor is bounded with k because the sequence $(\mathbf{v}_k)$ is weakly convergent in $\mathscr{W}^p(\mathbb{R})$. Hence, the first term on the right side of (3.4.48) tends to zero as $k \to \infty$. The second term on the right side of (3.4.48) is just

$$\int_{-T}^{T} \left\langle A^\infty \big(\mathbf{v}_k(t) - \mathbf{u}(t)\big),\, \boldsymbol{\phi}(t) \right\rangle \mathrm{d}t = \left\langle A^{\infty\sharp}(\mathbf{v}_k - \mathbf{u}),\, \boldsymbol{\phi} \right\rangle, \quad (3.4.51)$$

which tends to zero as k tends to infinity because $A^{\infty\sharp}$ is bounded and linear, and hence is weakly sequentially continuous; recall that $(\mathbf{v}_k)$ converges weakly to $\mathbf{u}$. This proves (3.4.46). It remains to show that

$$\tau_{\xi_k} \mathcal{E} G^\sharp(\mathbf{u}_{n_k}) \xrightharpoonup{w} G^{\infty\sharp}(\mathbf{u}) \text{ in } \mathscr{X}^p(\mathbb{R}) \text{ as } k \to \infty. \quad (3.4.52)$$

Because of the convergence of $\tau_{\xi_k}\mathcal{E}G$ to G^∞ is that of uniform convergence on the compact subsets of $\mathbb{R} \times \Omega$, we bring in the measurable selections $u_{n_k} = J\mathbf{u}_{n_k}$, $u = J\mathbf{u}$, and $\phi = J\boldsymbol{\phi}$, for a given $\boldsymbol{\phi} \in C_0^\infty(\mathbb{R},\, X^{p'})$. Once more, let $T > 0$ be such that $\operatorname{supp}\boldsymbol{\phi} \subset [-T,\, T]$, and let k be sufficiently large that $-T + \xi_k > 0$. We then can express the action of $\boldsymbol{\phi}$ on $\tau_{\xi_k}\mathcal{E}G^\sharp(\mathbf{u}_{n_k}) - G^{\infty\sharp}(\mathbf{u})$ as

$$\begin{aligned}
&\int_{\mathbb{R}} \big\langle \big(\tau_{\xi_k}\mathcal{E}G^\sharp(\mathbf{u}_{n_k}) - G^{\infty\sharp}(\mathbf{u})\big)(t),\, \boldsymbol{\phi}(t)\big\rangle \,\mathrm{d}t \\
&\quad= \int_{-T}^{T}\int_{\Omega} \Big(G\big(t+\xi_k,\, u_{n_k}(t+\xi_k,\, x)\big) - G^\infty\big(t,\, u(t,\, x)\big)\Big)\phi(t,\, x)\,\mathrm{d}x\,\mathrm{d}t \qquad (3.4.53)\\
&\quad= \int_{-T}^{T}\int_{\Omega} \Big(G\big(t+\xi_k,\, u_{n_k}(t+\xi_k,\, x)\big) - G^\infty\big(t,\, u_{n_k}(t+\xi_k,\, x)\big)\Big)\phi(t,\, x)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad + \int_{-T}^{T}\int_{\Omega} \Big(G^\infty\big(t,\, u_{n_k}(t+\xi_k,\, x)\big) - G^\infty\big(t,\, u(t,\, x)\big)\Big)\phi(t,\, x)\,\mathrm{d}x\,\mathrm{d}t \qquad (3.4.54)\\
&\quad= \int_{-T}^{T}\int_{\Omega} \Big(\big(\tau_{\xi_k}\mathcal{E}G\big)\big(t,\, u_{n_k}(t+\xi_k,\, x)\big) - G^\infty\big(t,\, u_{n_k}(t+\xi_k,\, x)\big)\Big)\phi(t,\, x)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad + \int_{\mathbb{R}} \big\langle \big(G^{\infty\sharp}(\mathbf{v}_k) - G^{\infty\sharp}(\mathbf{u})\big)(t),\, \boldsymbol{\phi}(t)\big\rangle\,\mathrm{d}t. \qquad (3.4.55)
\end{aligned}$$

The first term on the right side of (3.4.55) tends to zero as k tends to infinity, because of the uniform convergence of $\big(\tau_{\xi_k}\mathcal{E}G\big)$ to G^∞ on compact sets, and in particular on the product of $[-T,\, T]$ with a compact interval that contains the ranges of all of the functions u_n. The second term converges to zero because of the weak sequential continuity of $G^{\infty\sharp}$; see Lemma 3.2.6. (See also Lemma 2.2.7 on page 29 to see that $G^{\infty\sharp}$ satisfies the hypotheses of Lemma 3.2.6.)

This completes the verification that $\mathbf{h}$ is the weak limit, in $\mathscr{X}^p(\mathbb{R})$, of the sequence $(\mathbf{h}_k)$. However, $\mathbf{0}$ is also a weak limit of the sequence $(\mathbf{h_k}) = (\tau_{\xi_k}\mathcal{E}\mathbf{f}_{n_k})$, since $\xi_k \to \infty$, and each of the functions $\mathcal{E}\mathbf{f}_n$ are supported in $[-1,\, \infty)$. Hence, $\mathbf{h} = \mathbf{0}$ by the uniqueness of weak limits. According to the definition of $\mathbf{h}$, this means that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} - A^{\infty\sharp}\mathbf{u} + G^{\infty\sharp}(\mathbf{u}) = \mathbf{0}. \qquad (3.4.56)$$

Because $\omega(G)$ is assumed to be admissible, this implies that $\mathbf{u} = \mathbf{0}$, which completes the verification of the hypotheses of Lemma 3.4.5. According to Lemma 3.4.5, we conclude that

$\mathcal{H}$ is relatively compact in $C_{\{0\}}\big([0,\,\infty), X^p\big)$. Hence, there is a subsequence of $(\mathbf{u}_n)$ that converges in $C_{\{0\}}\big([0,\,\infty), L^p(\Omega)\big)$, and we are finished with Step 1.

STEP 2. We show that there is a subsequence of $(\mathbf{u}_n)$ that converges in the space $C_{\{0\}}\big([0,\,\infty), C_b(\overline{\Omega})\big)$. According to the result of Step 1, it is no loss of generality to assume that $(\mathbf{u}_n)$ converges in $C_{\{0\}}\big([0,\,\infty), L^p(\Omega)\big)$ to some $\mathbf{u}$. In particular,

$$\big\|\mathbf{u}_n(t) - \mathbf{u}(t)\big\|_{L^p(\Omega)} \to 0 \text{ as } n \to \infty,\ \text{uniformly in } t. \tag{3.4.57}$$

Since the sequence $(\mathbf{u}_n - \mathbf{u})$ is bounded in $\mathscr{W}^p\big([0,\,\infty)\big)$, Lemma 3.1.17 implies that

$$\big\|\mathbf{u}_n(t) - \mathbf{u}(t)\big\|_{\infty} \to 0 \text{ as } n \to \infty,\ \text{uniformly in } t, \tag{3.4.58}$$

which proves the desired convergence of $(\mathbf{u}_n)$ to $\mathbf{u}$ in $C_{\{0\}}\big([0,\,\infty), C_b(\overline{\Omega})\big)$.

STEP 3. We may now assume with no loss of generality that $(\mathbf{u}_n)$ converges to $\mathbf{u}$ in $C_{\{0\}}\big([0,\,\infty), C_b(\overline{\Omega})\big)$. We show that from this it follows that $\big(G^\sharp(\mathbf{u}_n)\big)$ converges to $G^\sharp(\mathbf{u})$ in $\mathscr{X}^p$. To do so, we use $u_n := J\mathbf{u}_n$ and $u := J\mathbf{u}$. For all $t > 0$, $x \in \Omega$, and $n \in \mathbb{N}$,

$$\begin{aligned}
&G\big(t,\, u_n(t,\, x)\big) - G\big(t,\, u(t,\, x)\big)\\
&\quad = \left(\int_0^1 D_\xi G\big(t,\, s u_n(t,\, x)\big)\,\mathrm{d}s\right) u_n(t,\, x) - \left(\int_0^1 D_\xi G\big(t,\, s u(t,\, x)\big)\,\mathrm{d}s\right) u(t,\, x) &(3.4.59)\\
&\quad = \left(\int_0^1 D_\xi G\big(t,\, s u_n(t,\, x)\big) - D_\xi G\big(t,\, s u(t,\, x)\big)\,\mathrm{d}s\right) u_n(t,\, x)\\
&\qquad\qquad + \left(\int_0^1 D_\xi G\big(t,\, s u(t,\, x)\big)\,\mathrm{d}s\right) \big((u_n(t,\, x) - u(t,\, x)\big). &(3.4.60)
\end{aligned}$$

For the first term on the right hand side of (3.4.60), we use the uniform convergence of (u_n) to u. Letting K be a compact interval that contains the range of each function u_n, the uniform continuity of $D_\xi G$ on $[0,\,\infty) \times K$ forces the integral in the first term on the right side of (3.4.60) to tend to zero, uniformly in $(t,\, x)$. The sequence (u_n) is bounded in

$\widehat{\mathscr{W}^p}\big([0,\,\infty)\big)$, and hence is bounded in $L^p\big([0,\,\infty)\times\Omega\big)$. Altogether, the first term on the right side of (3.4.60) tends to zero in $L^p\big([0,\,\infty)\times\Omega\big)$ as $n\to\infty$.

For the second term on the right side of (3.4.60), for each $t>0$, we consider pointwise multiplication by $\int_0^1 D_\xi G\big(t,\,su(t,\,x)\big)\,\mathrm{d}s$ as a linear operator $H(t)$ defined on $X^p = L^p(\Omega)$. For each $T>0$, put

$$M(T) := \max\left\{\big|D_\xi G(t,\,\xi)\big| : t\geq 0,\, \big|\xi\big|\leq T\right\}. \tag{3.4.61}$$

According to assumptions (3.2.22b) and (3.2.22c) from page 134, we have both

$$M(T)<\infty,\,\forall T>0, \tag{3.4.62}$$

and

$$\lim_{T\to 0} M(T) = 0. \tag{3.4.63}$$

Now let $g\in L^p(\Omega)$ be given. We have

$$\big\|H(t)g\big\|^p_{L^p(\Omega)} = \int_\Omega \left|\int_0^1 D_\xi G\big(t,\,su(t,\,x)\big)\,\mathrm{d}s g(x)\right|^p \mathrm{d}x \tag{3.4.64}$$

$$\leq \int_\Omega \big|M\big(\big\|u(t,\,\cdot)\big\|_\infty\big)g(x)\big|^p\,\mathrm{d}x, \tag{3.4.65}$$

so that

$$\big\|H(t)g\big\|_{L^p(\Omega)} \leq M\big(\big\|u(t,\,\cdot)\big\|_\infty\big)\,\big\|g\big\|_{L^p(\Omega)}. \tag{3.4.66}$$

According to Lemma 3.1.17,

$$\lim_{t\to\infty}\big\|u(t,\,\cdot)\big\|_\infty = 0. \tag{3.4.67}$$

Hence, multiplication by $H(t)$ is, for each $t\geq 0$, a continuous linear operator on $L^p(\Omega)$, and moreover

$$\lim_{t\to\infty}\big\|H(t)\big\|_{\mathrm{op}} = 0. \tag{3.4.68}$$

According to Lemma 3.4.7, pointwise multiplication by H is therefore a compact operator from $\mathscr{W}^p\big([0,\,\infty)\big)$ into $\mathscr{X}^p\big([0,\,\infty)\big)$. In particular, the "pointwise multiplication by H" operator transforms the sequence $(\mathbf{u}_n)$, which is weakly convergent in $\mathscr{W}^p\big([0,\,\infty)\big)$, into a sequence that is norm convergent in $\mathscr{X}^p\big([0,\,\infty)\big)$. Via the isometry J, all of this implies that the second term on the right side of (3.4.60) tends to zero in $L^p\big([0,\,\infty)\times\Omega\big)$ as $n\to\infty$. Hence,

$$\int_{\mathbb{R}\times\Omega}\big|G\big(t,\,u_n(t,\,x)\big)-G\big(t,\,u(t,\,x)\big)\big|^p\,\mathrm{d}(t,\,x)\to 0 \tag{3.4.69}$$

as $n\to\infty$, from which we conclude that $G^\sharp(\mathbf{u}_n)\to G^\sharp(\mathbf{u})$ in $\mathscr{X}^p\big([0,\,\infty)\big)$ as $n\to\infty$.

Step 4. Because of the convergence proved in the preceding step, the assumptions prior to the first step imply that

$$\left(\frac{\mathrm{d}}{\mathrm{dt}}-A^\sharp\right)\mathbf{u}_n=\mathbf{f}_n-G^\sharp(\mathbf{u}_n)\to\mathbf{f}-G^\sharp(\mathbf{u}), \tag{3.4.70}$$

in $\mathscr{X}^p\big([0,\,\infty)\big)$ as $n\to\infty$, and we still have the assumption that $(\mathbf{u}_n(0))$ converges to some g in Y^p. The continuous linear operator $\big(\frac{\mathrm{d}}{\mathrm{dt}}-A^\sharp\big)$ is assumed to be an isomorphism from $\mathscr{W}_0^p\big([0,\,\infty)\big)$ onto $\mathscr{X}^p\big([0,\,\infty)\big)$, and according to Lemma 3.3.7 this implies that $\big(\frac{\mathrm{d}}{\mathrm{dt}}-A^\sharp,\,E_0\big)$ is an isomorphism from $\mathscr{W}^p\big([0,\,\infty)\big)$ onto $\mathscr{X}^p\big([0,\,\infty)\big)\times Y^p$. According to Yood's criterion (see Property 1.4.9 on page 12), linear Fredholm operators are proper on closed bounded subsets. This properness on closed bounded subsets, the convergence of the sequence $\big(\big(\frac{\mathrm{d}}{\mathrm{dt}}+A^\sharp\big)\,\mathbf{u}_n,\,\mathbf{u}_n(0)\big)$, and the boundedness of $(\mathbf{u}_n)$ in $\mathscr{W}^p\big([0,\,\infty)\big)$ imply that $(\mathbf{u}_n)$ does indeed have a subsequence that is norm convergent in $\mathscr{W}^p\big([0,\,\infty)\big)$. This proves that $\big(\Phi_{A,G},\,E_0\big)$ is proper on the closed bounded subsets of $\mathscr{W}^p\big([0,\,\infty)\big)$. □

3.5 AN EXISTENCE THEOREM

Recall that we have a standing assumption that Ω is a bounded domain in $\mathbb{R}^d$, and that p is a real number greater than $d+1$. To keep the statement of the theorem uncrowded, we introduce the following condition separately, by way of a definition.

Definition 3.5.1. Given A and G, the pair $(f,\, g) \in L^p\big([0,\,\infty) \times \Omega\big) \times Y^p$ is said to satisfy the *a priori* bound condition if there exist a constant $C = C(f,\, g)$ and a C^1 path $(p,\, q) = (p_s,\, q_s) : [0,\, 1] \to L^p\big([0,\,\infty) \times \Omega\big) \times Y^p$ such that $(p_0,\, q_0) = (0,\, 0)$, and $(p_1,\, q_1) = (f,\, g)$, and such that for all $s \in [0,\, 1]$, each solution $u \in \widehat{\mathscr{W}^p}\big([0,\,\infty)\big)$ to the initial value problem

$$u_t(t,\, x) - A(t)u(t,\, x) - G\big(t,\, u(t,\, x)\big) = p_s(t,\, x), \quad t \geq 0,\, x \in \Omega; \tag{3.5.1a}$$

$$u(0,\, x) = q_s(x), \qquad x \in \Omega; \tag{3.5.1b}$$

satisfies the bound $\|u\|_{\widehat{\mathscr{W}^p}} \leq C$.

Theorem 3.5.2. *Assume the following:*

1. *The three conditions (3.2.21) hold of A.* *(See page 133)**, and the linear operator $\left(\frac{d}{dt} - A^\sharp\right)$ is an isomorphism of $\mathscr{W}_0^p\big([0,\,\infty)\big)$ onto $\mathscr{X}^p\big([0,\,\infty)\big)$.*
2. *The three conditions (3.2.22) hold of G.* *(See page 134)**, and G has an admissible omega-limit set $\omega(G)$.* *(See page 157).*
3. *For a given pair $(f,\, g) \in L^p\big([0,\,\infty) \times \Omega\big) \times Y^p$, the* a priori *bound condition is satisfied.*
4. *The only solution $u \in \widehat{\mathscr{W}^p}(I)$ to the homogeneous problem (use $f = 0$ and $g = 0$) associated with (3.0.1) on page 112 is $u = 0$.*

Then there is a solution $u \in \widehat{\mathscr{W}}^p\big([0,\,\infty)\big)$ *to the boundary value problem*

$$u_t(t,\,x) - A(t)u(t,\,x) + G\big(t,\,u(t,\,x)\big) = f(t,\,x), \quad t \ge 0,\, x \in \Omega; \tag{3.5.2a}$$

$$u(0,\,x) = g(x), \qquad x \in \Omega; \tag{3.5.2b}$$

$$u(t,\,x) = 0, \qquad t \ge 0,\, x \in \partial\Omega, \tag{3.5.2c}$$

$$\lim_{t\to\infty} \sup_{x\in\Omega} \big|u(t,\,x)\big| = 0. \tag{3.5.2d}$$

Proof. To begin, recall that the conclusion of the theorem is equivalent to the existence of $\mathbf{u} \in \mathscr{W}^p\big([0,\,\infty)\big)$, where $\mathbf{u} = J^{-1}u$, such that

$$\big(\Phi_{A,G},\,E_0\big)(\mathbf{u}) = (\mathbf{f},\,g), \tag{3.5.3}$$

where $\mathbf{f} = J^{-1}f$. Put

$$\Psi := \big(\Phi_{A,G},\,E_0\big), \tag{3.5.4}$$

$$X := \mathscr{W}^p\big([0,\,\infty)\big), \text{ and} \tag{3.5.5}$$

$$Y := \mathscr{X}^p\big([0,\,\infty)\big) \times Y^p. \tag{3.5.6}$$

According to Theorem 3.2.15 on page 141 (and Lemma 3.2.16), Ψ is a C^1 map of X into Y. Since $D\Psi(\mathbf{0}) = \big(\frac{\mathrm{d}}{\mathrm{dt}} - A^\sharp,\,E_0\big)$, it follows from hypothesis 1 and Theorem 3.3.8 on page 149 that Ψ is Fredholm of index zero from X into Y.

Let B be the open ball of radius $C+1$ centered at $\mathbf{0} \in X$, where C is the constant of the *a priori* bound condition, above. Hypothesis 2, with Theorem 3.4.9 on page 157, imply that Ψ is proper on $\overline{B}$. Also, the above *a priori* bound condition implies that

$$(\mathbf{p}_s,\,q_s) \in Y \setminus \Psi(\partial B), \tag{3.5.7}$$

for all $s \in [0,\, 1]$ (where $\mathbf{p}_s = J^{-1}p_s$). Thus, for all $s \in [0,\, 1]$

$$\bigl(\Psi,\, B,\, (\mathbf{p}_s,\, q_s)\bigr) \in \Xi, \tag{3.5.8}$$

as defined in Definition 1.5.1 on page 13. Accordingly, the absolute degree $\bigl|d\bigr| \bigl(\Psi,\, B,\, (\mathbf{p}_s,\, q_s)\bigr)$ is well-defined for all $s \in [0,\, 1]$. We introduce the following homotopy $h\colon [0,\, 1] \times X \to Y$:

$$h(s,\, \mathbf{u}) := \Psi(\mathbf{u}) - (\mathbf{p}_s,\, q_s). \tag{3.5.9}$$

Notice that $h(0,\, \cdot) = \Psi$, that $h(1,\, \cdot) = \Psi - (\mathbf{f},\, g)$, and that

$$h(s,\, \mathbf{u}) = 0 \qquad \Longleftrightarrow \qquad \Psi(\mathbf{u}) = (\mathbf{p}_s,\, q_s). \tag{3.5.10}$$

That h is C^1 follows trivially from the fact that Ψ is C^1. The properness of $h\bigr|_{[0,\,1]\times\overline{B}}$ results from the properness of Ψ on $\overline{B}$ as follows. Assume that $(s_n,\, \mathbf{u}_n)$ is a sequence in $[0,\, 1] \times \overline{B}$ such that $\bigl(h(s_n,\, u_n)\bigr)$ is convergent in Y, to some $(\mathbf{v},\, w)$. In any case, (s_n) has a convergent subsequence $s_{n_k} \to s_0 \in [0,\, 1]$. Thus,

$$\Psi(\mathbf{u}_{n_k}) = h(s_{n_k},\, \mathbf{u}_{n_k}) + (\mathbf{p}_{s_{n_k}},\, q_{s_{n_k}}) \tag{3.5.11}$$

$$\to (\mathbf{v},\, w) + (\mathbf{p}_{s_0},\, q_{s_0}) \text{ as } k \to \infty. \tag{3.5.12}$$

The already established properness of Ψ on $\overline{B}$ then implies that there is a convergent subsequence of $\bigl(u_{n_k}\bigr)$. This shows that $h\bigr|_{[0,\,1]\times\overline{B}}$ is proper. To see that h is Fredholm of index 1 from $[0,\, 1] \times X$ into Y, write Dh in the block matrix form

$$Dh(s,\, \mathbf{u}) = \Bigl((\dot{\mathbf{p}}_s \quad \dot{q}_s)^T \;\Big|\; D\Psi(\mathbf{u}) \Bigr), \tag{3.5.13}$$

which is a rank one perturbation of

$$L = \Bigl((\mathbf{0} \quad 0)^T \;\Big|\; D\Psi(\mathbf{u}) \Bigr). \tag{3.5.14}$$

Since L has the same target space and range as $D\Psi(\mathbf{u})$, and $\ker L = \mathbb{R} \times \ker D\Psi(\mathbf{u})$, the linear map L is Fredholm of index 1. Therefore, the compact perturbation $Dh(s,\, \mathbf{u})$ of L is also Fredholm of index 1. All of this implies that we may use the homotopy invariance of the absolute degree (Property 1.5.4 on page 13) to conclude that

$$\left|d\right| \big(h(0,\, \cdot),\, B,\, (0,0)\big) = \left|d\right| \big(h(s,\, \cdot),\, B,\, (0,0)\big) = \left|d\right| \big(h(1,\, \cdot),\, B,\, (0,0)\big) \tag{3.5.15}$$

for all $s \in [0,\, 1]$. As we have already noted, $h(0,\, \cdot) = \Psi$. Together, hypotheses 1 and 4 imply by Property 1.5.6 on page 14 that $\left|d\right| \big(\Psi,\, B,\, (\mathbf{0}, 0)\big) \neq 0$, so that also

$$\left|d\right| \big(\Psi - (\mathbf{f},\, g),\, B,\, (\mathbf{0}, 0)\big) \neq 0. \tag{3.5.16}$$

Because of the normalization property of the absolute degree (Property 1.5.2 on page 13), this implies that there is some $\mathbf{u} \in B \subset X = \mathscr{W}^p\big([0,\, \infty)\big)$ such that

$$\Psi(\mathbf{u}) = (\mathbf{f},\, g). \tag{3.5.17}$$

This is the same as the desired equation (3.5.3), and the proof is complete. $\square$

3.6 EXAMPLE

We now consider a more particular problem in order to show how the various hypotheses of the theorem can be met in practice. Especially, the techniques used to find *a priori* bounds should be expected to vary from problem to problem. We will take $A(t)$ to simply be the Laplacian

$$A(t) := \Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}\,. \tag{3.6.1}$$

Since A is autonomous, the three conditions (3.2.21) on page 133 are easily seen to hold of A. That $\left(\frac{\mathrm{d}}{\mathrm{dt}} - \Delta\right)$ is an isomorphism of $\mathscr{W}_0^p\big([0,\,\infty)\big)$ onto $\mathscr{X}^p\big([0,\,\infty)\big)$ follows from Corollary 8.5 in Rabier [Rab04b]; see Remark 3.3.3 on page 145. Note that at this point other choices could be made, including nonautonomous functions, but the Laplacian will be convenient for our derivation of *a priori* bounds. For G, we take any $G\colon [0,\,\infty) \times \mathbb{R} \to \mathbb{R}$ that satisfies the three conditions (3.2.22) on page 134 with $I = [0,\,\infty)$, and such that also

$$G(t,\,\xi)\xi \geq 0, \quad \forall t \geq 0,\, \forall \xi \in \mathbb{R}. \tag{3.6.2}$$

We also require that there are some $R > 0$ and $M > 0$ such that $\big|G(t,\,\xi)\big| > R$ for all $t \geq 0$ and $\big|\xi\big| \geq M$. Put another way,

$$\liminf_{\xi\ \to\infty} \inf_{t\geq 0} \big|G(t,\,\xi)\big| > R. \tag{3.6.3}$$

Of course, this condition is satisfied for all $R > 0$ if $\big|G(t,\,\xi)\big| \to \infty$ as $\big|\xi\big| \to \infty$ uniformly in $t \geq 0$. For example, if $\phi = \phi(t)$ is bounded, uniformly continuous, and nonnegative on $[0,\,\infty)$ and if $\epsilon > 0$, then $G(t,\,\xi) = \big(\phi(t) + \epsilon\big)\xi^{2k+1}$ satisfies all of the mentioned conditions, for each choice of $k \in \mathbb{N}$. Any (finite) convex combination of such functions also meets all of the conditions. To apply Theorem 3.5.2, we still need to verify the third and fourth hypotheses, and also the part of the second hypothesis that concerns the omega-limit set of G. In fact, we will verify these hypotheses almost simultaneously. Notice that if $C(0, 0) = 0$ in the *a priori* bound condition, then the uniqueness of the trivial solution to the homogeneous problem (the fourth hypothesis) follows. We can even check the admissibility of the omega-limit set of G in this way after checking that each member of $\omega(G)$ inherits all of the relevant properties from G. To find the desired bounds, we will first derive bounds for the L^p and L^∞ norms of solutions to (3.5.1). This will allow us to use the fact that $\left(\frac{\mathrm{d}}{\mathrm{dt}} - \Delta\right)$ is an isomorphism of $\mathscr{W}^p\big([0, \infty)\big)$ onto $\mathscr{X}^p\big([0, \infty)\big) \times Y^p$ to derive from the equation $\left(\frac{\mathrm{d}}{\mathrm{dt}} - \Delta\right) \mathbf{u} = \mathbf{f} - G^\sharp(\mathbf{u})$ the desired bound in $\mathscr{W}^p$ norm.

We begin with bounds in L^p. We will make use of the following two lemmas. The first is a version of Poincaré's inequality for $p > 2$, and the second is an integration by parts formula.

Lemma 3.6.1. *Because $p > d + 1 \geq 2$, there is a constant $C = C(p, vol(\Omega))$ such that for all $u \in W_0^{1,p}(\Omega)$,*

$$\int_\Omega |u|^p \, dx \leq C \int_\Omega |u|^{p-2} |\nabla u|^2 \, dx. \tag{3.6.4}$$

Proof. Since we are working with functions in $W_0^{1,p}(\Omega)$, it is no loss of generality to suppose that Ω is enlarged to a rectangle, say $\Omega = \prod_{i=1}^d (\alpha_i, \beta_i)$. Let $u \in W_0^{1,p}(\Omega)$, and fix a point $x = (x_1, x_2, \ldots, x_d) \in \Omega$.

Take $\eta = \eta(x) = (x_2, x_3, \ldots, x_d) \in \mathbb{R}^{d-1}$. We then write $x = (x_1, \eta)$. Since $(\alpha_1, \eta) \in \partial\Omega$, one has $u(\alpha_1,\, \eta) = 0$, and so

$$\left|u(x)\right|^p = \int_{\alpha_1}^{x_1} \frac{\partial}{\partial t} \left|u(t,\, \eta)\right|^p \mathrm{d}t \tag{3.6.5}$$

$$= \int_{\alpha_1}^{x_1} p \left|u(t,\, \eta)\right|^{p-1} \operatorname{sign} u(t,\, \eta) \frac{\partial u}{\partial x_1}(t,\, \eta)\, \mathrm{d}t \tag{3.6.6}$$

$$\leq p \int_{\alpha_1}^{\beta_1} \left|u(t,\, \eta)\right|^{p-1} \left|\frac{\partial u}{\partial x_1}(t,\, \eta)\right| \mathrm{d}t. \tag{3.6.7}$$

We now let x_1 vary between α_1 and β_1, and integrate, obtaining

$$\int_{\alpha_1}^{\beta_1} \left|u(x_1,\, \eta)\right|^p \mathrm{d}x_1 \leq p(\beta_1 - \alpha_1) \int_{\alpha_1}^{\beta_1} \left|u(x_1,\, \eta)\right|^{p-1} \left|\frac{\partial u}{\partial x_1}(x_1,\, \eta)\right| \mathrm{d}x_1. \tag{3.6.8}$$

Using Fubini's Theorem, we integrate with respect to $\eta \in \prod_{i=2}^d [\alpha_i,\, \beta_i]$ to obtain integrals over all of Ω.

$$\int_\Omega \left|u(x)\right|^p \mathrm{d}x \leq p(\beta_1 - \alpha_1) \int_\Omega \left|u(x)\right|^{p-1} \left|\frac{\partial u}{\partial x_1}\right| \mathrm{d}x. \tag{3.6.9}$$

Note that

$$\left|u(x)\right|^{p-1} \left|\frac{\partial u}{\partial x_1}\right| = \left|u(x)\right|^{p/2} \left|u(x)\right|^{p/2-1} \left|\frac{\partial u}{\partial x_1}\right|. \tag{3.6.10}$$

The factor $\left|u\right|^{p/2}$ is in L^2 because $u \in L^p$. We claim that the remaining factor of $\left|u\right|^{p/2-1} \left|\frac{\partial u}{\partial x_1}\right|$ is also in L^2. Of course, this is true if and only if $\left|u\right|^{p-2} \left|\frac{\partial u}{\partial x_1}\right|^2$ is in L^1. Because $\left|u\right|^{p-2}$ is in $L^{p/(p-2)}$ and $\left|\frac{\partial u}{\partial x_1}\right|^2$ is in $L^{p/2}$, the desired conclusion is reached by Hölder's inequality. Altogether, the Cauchy-Schwartz inequality applies to the right side of (3.6.9), resulting in the inequality

$$\int_\Omega \left|u(x)\right|^p \mathrm{d}x \leq p(\beta_1 - \alpha_1) \left(\int_\Omega \left|u(x)\right|^p \mathrm{d}x\right)^{1/2} \left(\int_\Omega \left|u\right|^{p-2} \left|\frac{\partial u}{\partial x_1}\right|^2 \mathrm{d}x\right)^{1/2}. \tag{3.6.11}$$

We square both sides and then multiply by $\left(\int_\Omega \left|u(x)\right|^p \mathrm{d}x\right)^{-1}$ to obtain

$$\int_\Omega \left|u(x)\right|^p \mathrm{d}x \leq p^2(\beta_1 - \alpha_1)^2 \int_\Omega \left|u\right|^{p-2} \left|\frac{\partial u}{\partial x_1}\right|^2 \mathrm{d}x. \tag{3.6.12}$$

Of course $\left|\nabla u\right| \geq \left|\frac{\partial u}{\partial x_1}\right|$, so the advertised result follows. $\square$

Remark 3.6.2. The proof is similar to that of the classical Poincaré inequality. Also, by application of Hölder's inequality, one obtains

$$\left\|u\right\|_p \leq C \left\|\nabla u\right\|_p \tag{3.6.13}$$

as a corollary to Lemma 3.6.1. We would also like to point out, as shared with us by Professor Manfredi, that Lemma 3.6.1 can also be proved by application of the classical Poincaré inequality to the function $\left|u\right|^{p/2}$, which is easily seen to be in $W_0^{1,2}(\Omega)$. ◊

Lemma 3.6.3. *Suppose that $u \in W^{1,p}(\Omega)$ and that $v \in W_0^{1,p'}(\Omega)$, where $1/p + 1/p' = 1$. Then*

$$\int_\Omega \frac{\partial u}{\partial x_i} v \, dx = -\int_\Omega u \frac{\partial v}{\partial x_i} \, dx. \tag{3.6.14}$$

In particular, if u is also in $W^{2,p}(\Omega)$, then

$$\int_\Omega (\Delta u) v \, dx = -\int_\Omega (\nabla u) \cdot (\nabla v) \, dx. \tag{3.6.15}$$

Proof. Equation (3.6.14) is true by definition, in case $v \in C_0^\infty(\Omega)$ is a test function. Otherwise, we just take a sequence (ϕ_n) of test functions that converges to v in $W_0^{1,p'}(\Omega)$. Then, letting $n \to \infty$ in the equation

$$\int_\Omega \frac{\partial u}{\partial x_i} \phi_n \, \mathrm{d}x = -\int_\Omega u \frac{\partial \phi_n}{\partial x_i} \, \mathrm{d}x \tag{3.6.16}$$

gives (3.6.15). Equation (3.6.15) follows at once, by using (3.6.14) with u replaced by each of its first order partial derivatives:

$$\int_\Omega (\Delta u) v \, \mathrm{d}x = \sum_{i=1}^d \int_\Omega \frac{\partial^2 u}{\partial x_i^2} v \, \mathrm{d}x \tag{3.6.17}$$

$$= \sum_{i=1}^d - \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, \mathrm{d}x \tag{3.6.18}$$

$$= -\int_\Omega (\nabla u) \cdot (\nabla v) \, \mathrm{d}x. \tag{3.6.19}$$

$\square$

Lemma 3.6.4. *Let $R > 0$. There exists a constant $C_p = C_p(R) > 0$ such that*

$$\left\|u\right\|_p \leq C_p \tag{3.6.20}$$

whenever $u \in \widehat{\mathscr{W}^p}\big([0,\,\infty)\big)$ solves the initial value problem

$$u_t - \Delta u + \widetilde{G}(u) = f; \tag{3.6.21}$$

$$u(0,\,\cdot) = g, \tag{3.6.22}$$

for some f and g such that

$$f \in L^p\big([0,\,\infty) \times \Omega\big), \qquad \left\|f\right\|_p \leq R; \tag{3.6.23}$$

$$g \in Y^p, \qquad \left\|g\right\|_{Y^p} \leq R. \tag{3.6.24}$$

Proof. The first part of the argument (until we integrate with respect to t) is understood to hold for all $t \geq 0$, except possibly for some t belonging to a set of measure zero. We begin by multiplying both sides of (3.6.21) by $u\left|u\right|^{p-2} \in L^{p'}$ and integrating over Ω:

$$\int_\Omega \frac{\partial u}{\partial t} u\left|u\right|^{p-2} \mathrm{d}x - \int_\Omega \Delta u u\left|u\right|^{p-2} \mathrm{d}x + \int_\Omega \widetilde{G}(u) u\left|u\right|^{p-2} \mathrm{d}x = \int_\Omega f u\left|u\right|^{p-2} \mathrm{d}x. \tag{3.6.25}$$

By assumption (3.6.2) on G, the third term is non-negative, and hence the following inequality results:

$$\int_\Omega \frac{\partial u}{\partial t} u\left|u\right|^{p-2} \mathrm{d}x - \int_\Omega \Delta u u\left|u\right|^{p-2} \mathrm{d}x \leq \int_\Omega f u\left|u\right|^{p-2} \mathrm{d}x. \tag{3.6.26}$$

We now consider each term in(3.6.26) separately. For the first term, note that

$$\frac{\partial}{\partial t}\left|u\right|^p = p\left|u\right|^{p-1} \operatorname{sign} u \frac{\partial u}{\partial t} \tag{3.6.27}$$

$$= p\left|u\right|^{p-2} u \frac{\partial u}{\partial t}. \tag{3.6.28}$$

Hence, the first integral in (3.6.26) is

$$\int_\Omega \frac{\partial u}{\partial t} u \left|u\right|^{p-2} \mathrm{d}x = \frac{1}{p} \int_\Omega \frac{\partial}{\partial t} \left|u\right|^{p} \mathrm{d}x \tag{3.6.29}$$

$$= \frac{1}{p} \frac{\partial}{\partial t} \int_\Omega \left|u\right|^{p} \mathrm{d}x. \tag{3.6.30}$$

(The latter equality is proved directly, by using test functions.) For the second integral in (3.6.26), we use integration by parts (Lemma 3.6.3).

$$-\int_\Omega \Delta u u \left|u\right|^{p-2} \mathrm{d}x = \int_\Omega \nabla u \cdot \nabla \left(u \left|u\right|^{p-2}\right) \mathrm{d}x. \tag{3.6.31}$$

We calculate

$$\nabla \left(u \left|u\right|^{p-2}\right) = \nabla u \left|u\right|^{p-2} + u(p-2) \left|u\right|^{p-3} \operatorname{sign} u \nabla u \tag{3.6.32}$$

$$= \nabla u \left|u\right|^{p-2} + (p-2)\nabla u \left|u\right|^{p-2} \tag{3.6.33}$$

$$= (p-1)\nabla u \left|u\right|^{p-2}. \tag{3.6.34}$$

Hence,

$$-\int_\Omega \Delta u u \left|u\right|^{p-2} \mathrm{d}x = (p-2) \int_\Omega \left|\nabla u\right|^{2} \left|u\right|^{p-2} \mathrm{d}x. \tag{3.6.35}$$

As a result, the inequality (3.6.26) becomes

$$\frac{1}{p} \frac{\partial}{\partial t} \int_\Omega \left|u\right|^{p} \mathrm{d}x + (p-2) \int_\Omega \left|\nabla u\right|^{2} \left|u\right|^{p-2} \mathrm{d}x \leq \int_\Omega f u \left|u\right|^{p-2} \mathrm{d}x. \tag{3.6.36}$$

In light of Lemma 3.6.1, this implies that

$$\frac{1}{p} \frac{\partial}{\partial t} \int_\Omega \left|u\right|^{p} \mathrm{d}x + C_\Omega \int_\Omega \left|u\right|^{p} \mathrm{d}x \leq \int_\Omega f u \left|u\right|^{p-2} \mathrm{d}x, \tag{3.6.37}$$

where C_Ω is the constant from Lemma 3.6.1. This inequality holds for almost every $t \geq 0$, and we integrate on $[0, \infty)$. Because $\left\| u(t, \cdot) \right\|_{L^p(\Omega)} \to 0$ as $t \to \infty$, the first term in (3.6.36) results in

$$\int_0^\infty \frac{1}{p} \left(\frac{\partial}{\partial t} \int_\Omega \left| u \right|^p \mathrm{d}x \right) \mathrm{d}t = -\frac{1}{p} \left\| u(0, \cdot) \right\|_{L^p(\Omega)}^p = -\frac{1}{p} \left\| g \right\|_{L^p(\Omega)}^p . \tag{3.6.38}$$

The result of integrating the second term is

$$\int_0^\infty C_\Omega \int_\Omega \left| u \right|^p \mathrm{d}x \, \mathrm{d}t = C_\Omega \left\| u \right\|_p^p . \tag{3.6.39}$$

For the third term, we apply Hölder's inequality, which is applicable to the function pair $f \in L^p\big([0, \infty) \times \Omega\big)$ and $u \left| u \right|^{p-2} \in L^{p/(p-1)}\big([0, \infty) \times \Omega\big)$. The resulting inequality involving the right hand side of (3.6.36), upon integrating over $[0, \infty)$, is

$$\begin{aligned} \int_0^\infty \int_\Omega f u \left| u \right|^{p-2} \mathrm{d}x &\leq \left(\int_{[0, \infty) \times \Omega} \left| f \right|^p \mathrm{d}(t, x) \right)^{1/p} \left(\int_{[0, \infty) \times \Omega} \left| u \right|^p \mathrm{d}(t, x) \right)^{(p-1)/p} && (3.6.40) \\ &= \left\| f \right\|_p \left\| u \right\|_p^{p-1} && (3.6.41) \\ &\leq R \left\| u \right\|_p^{p-1} . && (3.6.42) \end{aligned}$$

Altogether, inequality (3.6.36) implies that

$$\left\| u \right\|_p^p \leq C_1 \left(R \left\| u \right\|_p^{p-1} + \left\| g \right\|_{L^p(\Omega)}^p \right), \tag{3.6.43}$$

where C_1 depends only on p and Ω. We divide both sides by $\left\| u \right\|_p^{p-1}$, to obtain

$$\left\| u \right\|_p \leq C_1 R + C_1 \frac{\left\| g \right\|_{L^p(\Omega)}^p}{\left\| u \right\|_p^{p-1}}. \tag{3.6.44}$$

To account for the second term on the right side, notice that

$$\|g\|_{L^p(\Omega)} = \inf\left\{\|\mathbf{v}\|_{C_b\left([0,\infty),\, L^p(\Omega)\right)} : \mathbf{v}(0) = g\right\} \tag{3.6.45}$$

$$\le C_2 \inf\left\{\|\mathbf{v}\|_{\mathscr{W}^p\left([0,\infty)\right)} : \mathbf{v}(0) = g\right\} \tag{3.6.46}$$

$$= C_2 \|g\|_{Y^p}, \tag{3.6.47}$$

where C_2 is the constant of the embedding of $\mathscr{W}^p\big([0,\infty)\big)$ into $C_b\big([0,\infty),\, L^p(\Omega)\big)$. Hence, inequality (3.6.44) implies that

$$\|u\|_p \le C_1 R + C_1 \frac{(C_2 R)^p}{\|u\|_p^{p-1}}. \tag{3.6.48}$$

This implies that the choice $C_p = \max\big(1,\, C_1 R + C_1 (C_2 R)^p\big)$ works. □

For later use in verifying the second and fourth hypotheses of Theorem 3.5.2, we pause for the following corollaries to the proof of Lemma 3.6.4.

Corollary 3.6.5. *In the situation of Lemma 3.6.4, if $f = 0$ and if $u \in \widehat{\mathscr{W}}_0^p\big([0,\infty)\big)$ satisfies (3.6.21), then $u = 0$. Similarly, suppose for the moment that $[0,\infty)$ is replaced by $\mathbb{R}$; that is, the function $G\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfies the three conditions (3.2.22) on page 134 with $I = \mathbb{R}$, and G satisfies (3.6.2) for all $t \in \mathbb{R}$, and (3.6.3) is satisfied with "$\sup_{t\ge 0}$" replaced by "$\sup_{t\in\mathbb{R}}$". In this setting, if $f = 0$ and if $u \in \widehat{\mathscr{W}}^p(\mathbb{R})$ satisfies (3.6.21), then $u = 0$.*

Proof. For the first assertion, we return to the proof of Lemma 3.6.21, but with $R = 0$ and $u(0,x) = 0$. No changes are necessary, except that (3.6.43) becomes

$$\|u\|_p^p = 0, \tag{3.6.49}$$

as claimed. For the second assertion, we integrate (3.6.37) on $\mathbb{R}$ instead of just on $[0,\infty)$. This time, the first term vanishes altogether. The other terms are not changed, except of course that the resulting $p-$norms are now in $\mathbb{R} \times \Omega$. Hence, in place of (3.6.43), we once again obtain

$$\left\|u\right\|_p^p = 0. \tag{3.6.50}$$

$\square$

Before proceeding to obtain bounds in L^∞, we pause to explain the simple idea behind the rather technical argument. Suppose for the moment that u is smooth. The function u achieves its maximum value at some point $(t_0,\, x_0) \in [0,\, \infty) \times \Omega$, because $\left|u(t,\, x)\right| \to 0$ as $t \to \infty$ and because u vanishes on $[0,\, \infty) \times \partial\Omega$. If $t_0 = 0$, then we can easily bound $\left\|u\right\|_\infty$ by a constant that depends only on $\left\|g\right\|_{Y^p}$. Otherwise, $(t_0,\, x_0)$ lies in the open set $(0,\, \infty) \times \Omega$. Suppose that $u(t_0,\, x_0) > 0$; the argument is similar if $u(t_0,\, x_0) < 0$. Then

$$\frac{\partial u}{\partial t}(t_0,\, x_0) = 0, \tag{3.6.51}$$

and

$$-\Delta u(t_0,\, x_0) \geq 0. \tag{3.6.52}$$

Since u satisfies (3.6.21), we have

$$G\big(t_0,\, u(t_0,\, x_0)\big) \leq f(t_0,\, x_0). \tag{3.6.53}$$

Since $u(t_0,\, x_0) > 0$, we know from assumption (3.6.2) that $G\big(t_0,\, u(t_0,\, x_0)\big) \geq 0$. Thus,

$$\left|G\big(t_0,\, u(t_0,\, x_0)\big)\right| \leq \left\|f\right\|_\infty, \tag{3.6.54}$$

which we will assume is finite. If also $\left\|f\right\|_\infty$ is smaller than the constant R from assumption (3.6.3), then we have found an implicit bound for $u(t_0,\, x_0) = \left\|u\right\|_\infty$. Our goal is to remove the assumption that u is smooth. To begin, we collect some properties of convolution and smoothing. We bring in the standard mollifier $\eta : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ given by

$$\eta(t,\, x) := \begin{cases} \beta \exp\Big(\big(t^2 + \left|x\right|^2 - 1\big)^{-1}\Big) & \text{if } t^2 + \left|x\right|^2 < 1; \\ 0 & \text{otherwise,} \end{cases} \tag{3.6.55}$$

where $\beta > 0$ is chosen so that $\int \eta = 1$. It is a standard result that $\eta \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d)$. As usual we take, for each $\epsilon > 0$,

$$\eta_\epsilon(t, x) = \epsilon^{-(d+1)} \eta(t/\epsilon,\, x/\epsilon), \tag{3.6.56}$$

so that η_ϵ is supported in the ball of radius ϵ about the origin, and $\int \eta_\epsilon = 1$. We then define, for any locally integrable function $u : [0,\, \infty) \times \Omega \to \mathbb{R}$, its η_ϵ-mollification $u^{(\epsilon)} = \eta_\epsilon * u : (\epsilon,\, \infty) \times \Omega_\epsilon \to \mathbb{R}$, where

$$\Omega_\epsilon := \left\{ (t,\, x) \in \Omega \colon\ \operatorname{dist}\big((t,\, x),\, \partial\Omega\big) > \epsilon \right\}, \tag{3.6.57}$$

and

$$\big(\eta_\epsilon * u\big)(t,\, x) := \int_{[0,\, \infty) \times \Omega} \eta_\epsilon(t - s,\, x - y) u(s,\, y)\, \mathrm{d}(s,\, y). \tag{3.6.58}$$

We shall make use of the following results. These results are standard, and can be found in Evans [Eva98].[5]

Lemma 3.6.6. *With the above definitions we have the following properties.*

1. $u^{(\epsilon)} \in C^\infty\big((\epsilon, \infty) \times \Omega_\epsilon\big)$.
2. *If α is any multi-index*[6]*, then* $D^\alpha u^{(\epsilon)} = D^\alpha \eta_\epsilon * u$.

[5] Assertion (1) is Theorem 6, part (i) of Appendix C in Evans [Eva98], and assertion (2) is derived as the proof of the same. Assertion (3) is Theorem 6, part (iii) of Appendix C in Evans [Eva98].

[6] meaning that we are allowing D^α to include derivatives with respect to t.

3. If u is continuous on $[0,\,\infty)\times\Omega$, then $u^{(\epsilon)}\to u$ uniformly on each compact $K \subset [0,\,\infty) \times \Omega$.

We are ready to establish bounds in L^∞.

Lemma 3.6.7. *Let $R > 0$ be given to satisfy assumption (3.6.3). There exists a constant $C_\infty = C_\infty(R) > 0$ such that*

$$\|u\|_\infty \le C_\infty \tag{3.6.59}$$

whenever $u \in \widehat{\mathscr{W}^p}\big([0,\,\infty)\big)$ solves the initial value problem

$$u_t - \Delta u + \widetilde{G}(u) = f; \tag{3.6.60}$$

$$u(0,\,\cdot) = g, \tag{3.6.61}$$

for some f and g such that

$$f \in L^p\big([0,\,\infty) \times \Omega\big), \qquad \|f\|_\infty \le R; \tag{3.6.62}$$

$$g \in Y^p, \qquad \|g\|_{Y^p} \le R. \tag{3.6.63}$$

Proof. First, we consider the case that $\|u\|_\infty = \|g\|_\infty$; that is, we suppose that $|u|$ attains its maximum at $t = 0$. It is clear that in general

$$\|g\|_\infty \le \inf\left\{\|v\|_\infty : v \in \widehat{\mathscr{W}^p}\big([0,\,\infty)\big) \text{ and } v(0,\,\cdot) = g\right\}. \tag{3.6.64}$$

The choice $v = u$ shows that inequality (3.6.64) must in fact be an equality[7]. Hence,

$$\|u\|_\infty \le \inf\left\{\|v\|_\infty : v \in \widehat{\mathscr{W}^p}\big([0,\,\infty)\big) \text{ and } v(0,\,\cdot) = g\right\} \tag{3.6.65}$$

$$\le C \inf\left\{\|J^{-1}v\|_{\mathscr{W}^p\left([0,\,\infty)\right)} : v \in \widehat{\mathscr{W}^p}\big([0,\,\infty)\big) \text{ and } v(0,\,\cdot) = g\right\} \tag{3.6.66}$$

$$= C\,\|g\|_{Y^p} \tag{3.6.67}$$

$$\le CR, \tag{3.6.68}$$

[7]It is not hard to show that inequality (3.6.64) is an equality in general; multiply a given v by functions of the form e^{-nt}.

where the constant $C > 0$ is that of the embedding of $\widehat{\mathscr{W}^p}\big([0,\,\infty)\big)$ in $C^0\big([0,\,\infty)\times\Omega\big)$.

Otherwise, we suppose temporarily that $\|u\|_\infty$ is attained by u, rather than by $-u$. Let $u_n := \eta_{\delta_n} * u$ be a sequence of mollifications of u, where $\delta_n \to 0$ and $\Omega_{\delta_1} \neq \emptyset$. For each $n \in \mathbb{N}$, the function u_n is smooth on its domain $(\delta_n,\,\infty)\times\Omega_{\delta_n}$, according to Lemma 3.6.6. Also from Lemma 3.6.6, the sequence (u_n) converges uniformly on compact subsets of $(0,\,\infty)\times\Omega$. Recall that $u = 0$ on $[0,\,\infty)\times\partial\Omega$, $u(t,\,x)\to 0$ as $t\to\infty$, and u does not attain its maximum $\alpha := \|u\|_\infty$ at $t = 0$. Therefore, $K = u^{-1}(\{\alpha\})$ is a compact subset of $(0,\,\infty)\times\Omega$. Let V be an bounded open set such that $K \subset V \subset \overline{V} \subset (0,\,\infty)\times\Omega$. For each $n\in\mathbb{N}$, let $(t_n,\,x_n)$ be a point in $\overline{V}$ where $u_n\big|_{\overline{V}}$ attains its maximum value. By compactness, we may suppose with no loss of generality that the sequence $(t_n,\,x_n)$ converges to a point $(t_0,\,x_0)\in\overline{V}$. Since Lemma 3.6.6 ensures that (u_n) converges uniformly to u on the compact subset $\overline{V}$ of $(0,\,\infty)\times\Omega$, it follows that

$$u(t_0,\,x_0) = \lim_{n\to\infty} u_n(t_n,\,x_n) = \|u\|_\infty\,, \tag{3.6.69}$$

and hence that $(t_0,\,x_0)\in K\subset V$ is an interior point of $[0,\,\infty)\times\Omega$. For each $n\in\mathbb{N}$, take

$$\phi_n(t,\,x) := \eta_{\delta_n}(t_n - t,\,x_n - x). \tag{3.6.70}$$

For sufficiently large n, the point $(t_0,\,x_0)$ is in $(\delta_n,\,\infty)\times\Omega_{\delta_n}$, whence $\phi_n\in C_0^\infty\big([0,\,\infty)\times\Omega\big)$. Hence, we can multiply both sides of equation (3.6.60) by ϕ_n and integrate over $[0,\,\infty)\times\Omega$, obtaining four integrals:

$$\int u_t\phi_n - \int \Delta u\phi_n + \int \widetilde{G}(u)\phi_n = \int f\phi_n. \tag{3.6.71}$$

We consider each integral. First,

$$\int_{[0,\,\infty)\times\Omega} \frac{\partial}{\partial t} u(t,\,x)\phi_n(t,\,x)\,\mathrm{d}(t,\,x) = \int_{[0,\,\infty)\times\Omega} \frac{\partial}{\partial t} u(t,\,x)\eta_{\delta_n}(t_n - t,\,x_n - x)\,\mathrm{d}(t,\,x) \tag{3.6.72}$$

$$= \int_{[0,\,\infty)\times\Omega} u(t,\,x)\frac{\partial \eta_{\delta_n}}{\partial t}(t_n - t,\,x_n - x)\,\mathrm{d}(t,\,x) \tag{3.6.73}$$

$$= \left(\frac{\partial \eta_{\delta_n}}{\partial t} * u\right)(t_n,\,x_n) \tag{3.6.74}$$

$$= \frac{\partial u_n}{\partial t}(t_n,\,x_n) \tag{3.6.75}$$

$$= 0, \tag{3.6.76}$$

since u_n achieves an interior maximum at $(t_n,\,x_n)$; we have used assertion 2 of Lemma 3.6.6. In the same way, we find that

$$-\int_{[0,\,\infty)\times\Omega} \Delta u(t,\,x)\phi_n(t,\,x)\,\mathrm{d}(t,\,x) = -\Delta u_n(t_n,\,x_n) \geq 0. \tag{3.6.77}$$

For the third integral in (3.6.71), we have

$$\int_{[0,\,\infty)\times\Omega} \big(\widetilde{G}(u)\big)(t,\,x)\phi_n(t,\,x)\,\mathrm{d}(t,\,x) = \int_{[0,\,\infty)\times\Omega} \big(\widetilde{G}(u)\big)(t,\,x)\eta_{\delta_n}(t_n - t,\,x_n - x)\,\mathrm{d}(t,\,x) \tag{3.6.78}$$

$$= \left(\eta_{\delta_n} * \widetilde{G}(u)\right)(t_n,\,x_n). \tag{3.6.79}$$

Altogether, (3.6.71) implies that

$$\left(\eta_{\delta_n} * \widetilde{G}(u)\right)(t_n,\,x_n) \leq \int_{[0,\,\infty)\times\Omega} f(t,\,x)\eta_{\delta_n}(t_n - t,\,x_n - x)\,\mathrm{d}(t,\,x) \tag{3.6.80}$$

$$\leq \left\|f\right\|_\infty \tag{3.6.81}$$

$$\leq R. \tag{3.6.82}$$

Now as $n \to \infty$, the sequence $\left(\eta_{\delta_n} * \widetilde{G}(u)\right)$ converges uniformly on compact sets to $\widetilde{G}(u)$, since $\widetilde{G}(u)$ is continuous. Thus, since $(t_n,\, x_n)$ converges to $(t_0,\, x_0)$, we obtain

$$G\big(t_0,\, \|u\|_\infty\big) \leq R. \tag{3.6.83}$$

Recall the assumption that $G(t,\, \xi)\xi \geq 0$, which implies that $G(\big(t_0,\, \|u\|_\infty\big) \geq 0$. Also, recall the assumption that $R < \liminf_{\xi \ \to\infty} \inf_{t\geq 0} \big|G(t,\, \xi)\big|$. Hence, we have

$$\inf_{t\geq 0} \big|G(t,\, \|u\|_\infty)\big| \leq R < \liminf_{\xi \ \to\infty} \inf_{t\geq 0} \big|G(t,\, \xi)\big| \,. \tag{3.6.84}$$

This provides the existence of an upper bound $C = C(R)$ for $\|u\|_\infty$.

It remains to consider the case that $-u$ attains $\|u\|_\infty$. In this case, we need only replace $(t_n,\, x_n)$ with a sequence of minimizers for (u_n). This reverses the sign of $\Delta u_n(t_n,\, x_n)$, resulting in

$$\left(\eta_{\delta_n} * \widetilde{G}(u)\right)(t_n,\, x_n) \geq \int_{[0,\,\infty)\times\Omega} f(t,\, x)\eta_{\delta_n}(t_n - t,\, x_n - x)\, \mathrm{d}(t,\, x) \tag{3.6.85}$$

$$\geq -\|f\|_\infty \tag{3.6.86}$$

$$\geq -R. \tag{3.6.87}$$

In this case, the result of letting $n \to \infty$ is

$$G\big(t_0,\, -\|u\|_\infty\big) \geq -R. \tag{3.6.88}$$

Since $G\big(t_0,\, -\|u\|_\infty\big) \leq 0$, we have

$$\inf_{t\geq 0} \big|G(t,\, -\|u\|_\infty)\big| \leq R < \liminf_{\xi \ \to\infty} \inf_{t\geq 0} \big|G(t,\, \xi)\big| \,, \tag{3.6.89}$$

which again provides the existence of an upper bound $C = C(R)$ for $\|u\|_\infty$. $\square$

We are now prepared to derive the desired *a priori* bounds in $\widehat{\mathscr{W}^p}$ norm.

Theorem 3.6.8. *Let $R > 0$ be given to satisfy assumption (3.6.3). There exists a constant $C = C(R) > 0$ such that*

$$\|u\|_\infty \leq C \tag{3.6.90}$$

whenever $u \in \widehat{\mathscr{W}^p}\big([0,\,\infty)\big)$ solves the initial value problem

$$u_t - \Delta u + \widetilde{G}(u) = f; \tag{3.6.91}$$

$$u(0,\,\cdot) = g, \tag{3.6.92}$$

for some f and g such that

$$f \in L^p\big([0,\,\infty)\times\Omega\big), \qquad \|f\|_p + \|f\|_\infty \leq R; \tag{3.6.93}$$

$$g \in Y^p, \qquad \|g\|_{Y^p} \leq R. \tag{3.6.94}$$

Proof. The operator $\big(\frac{\partial}{\partial t} - \Delta\big)$ is an isomorphism of $\mathscr{W}_0^p\big([0,\,\infty)\big)$ onto the Banach space $\mathscr{X}^p\big([0,\,\infty)\big) = L^p\big([0,\,\infty),\,L^p(\Omega)\big)$. (For this, see Rabier ([Rab04b], Theorem 3.1, as well as the discussion in the final section of [Rab03]).) Thus, according to Lemma 3.3.7, the augmented operator $\big(\frac{\partial}{\partial t} - \Delta,\,E_0\big)$ is an isomorphism of $\mathscr{W}^p\big([0,\,\infty)\big)$ onto $\mathscr{X}^p\big([0,\,\infty)\big)\times Y^p$. Hence there exists a constant C_1 such that

$$\|u\|_{\widehat{\mathscr{W}^p}([0,\,\infty))} \leq C_1\big(\|u_t - \Delta u\|_{L^p([0,\,\infty)\times\Omega)} + \|g\|_{Y^p}\big) \tag{3.6.95}$$

$$\leq C_1\Big(\|f\|_{L^p([0,\,\infty)\times\Omega)} + \big\|\widetilde{G}(u)\big\|_{L^p([0,\,\infty)\times\Omega)} + \|g\|_{Y^p}\Big), \tag{3.6.96}$$

assuming that u satisfies (3.6.91)– (3.6.92). So we focus our attention on $\widetilde{G}(u)$. According to Lemma 3.6.7, we have $\|u\|_\infty < C_\infty$. Let $M > 0$ be a bound for $|D_\xi G|$ on $[0,\,\infty)\times[-C_\infty,\,C_\infty]$;

the existence of M is guaranteed by assumption (3.2.22b) on page 134. We can now estimate that

$$\big|G\big(t,\, u(t,\, x)\big)\big| = \left| \int_0^1 D_\xi G\big(t,\, su(t,\, x)\big)\, \mathrm{d}s\, u(t,\, x) \right| \tag{3.6.97}$$

$$\leq M \left|u(t,\, x)\right| . \tag{3.6.98}$$

Taking p^{th} powers and integrating over $[0,\, \infty) \times \Omega$ yields

$$\left\|\widetilde{G}(u)\right\|_{L^p\left([0,\,\infty)\times\Omega\right)} \leq M \left\|u\right\|_{L^p\left([0,\,\infty)\times\Omega\right)} . \tag{3.6.99}$$

According to Lemma 3.6.4, this implies that

$$\left\|\widetilde{G}(u)\right\|_{L^p\left([0,\,\infty)\times\Omega\right)} \leq MC_p. \tag{3.6.100}$$

With (3.6.96), this proves the desired result, with $C = C_1(2R + MC_p)$. □

We now show how to use Theorem 3.6.8 to satisfy the third hypothesis of Theorem 3.5.2. Suppose that $f \in L^p\big([0,\, \infty) \times \Omega\big) \cap L^\infty\big([0,\, \infty) \times \Omega\big)$ and $g \in Y^p$ are given. Take $(p,\, q)$ to be the linear path $(p_s,\, q_s) = (sf,\, sg)$. Then, for all $s \in [0,\, 1]$,

$$\left\|p_s\right\|_p + \left\|p_s\right\|_\infty \leq \left\|f\right\|_p + \left\|f\right\|_\infty ; \tag{3.6.101}$$

$$\left\|q_s\right\|_{Y^p} \leq \left\|g\right\|_{Y^p} . \tag{3.6.102}$$

Thus, as long as there is $R > 0$ that satisfies condition (3.6.3) and such that

$$\left\|f\right\|_p + \left\|f\right\|_\infty \leq R; \tag{3.6.103}$$

$$\left\|g\right\|_{Y^p} \leq R \tag{3.6.104}$$

then Theorem 3.6.8 gives the constant C required by the *a priori* bound condition in Definition 3.5.1. This satisfies the third hypothesis of Theorem 3.5.2.

To check the fourth hypothesis in Theorem 3.5.2, suppose that $u \in \widehat{\mathscr{W}^p}\big([0, \infty)\big)$ solves the homogeneous initial value problem

$$u_t - \Delta u + \widetilde{G}(u) = 0; \tag{3.6.105}$$

$$u(0, \cdot) = 0. \tag{3.6.106}$$

Then, according to Corollary 3.6.5, $u = 0$.

The only hypothesis of Theorem 3.5.2 that remains is that G should have an admissible omega-limit set. (This is part of the second hypothesis.) Let $G^\infty \in \omega(G)$ be given. Accordingly, let (σ_n) be a sequence in $[0, \infty)$ such that $\sigma_n \to \infty$ and

$$G^\infty = \operatorname*{CO\text{-}lim}_{n\to\infty} \tau_{\sigma_n} G. \tag{3.6.107}$$

It follows immediately that $G^\infty(t, \xi)\xi \geq 0$ by passing to the limit in (3.6.2), with t replaced by $t + \sigma_n$:

$$G^\infty(t, \xi)\xi = \lim_{n\to\infty} G(t + \sigma_n, \xi)\xi \geq 0, \quad \forall t \in \mathbb{R}, \forall \xi \in \mathbb{R}. \tag{3.6.108}$$

Also, for all $t \in \mathbb{R}$,

$$\left|G^\infty(t, \xi)\right| = \lim_{n\to\infty} \left|G(t + \sigma_n, \xi)\right| \tag{3.6.109}$$

$$\geq \inf_{s\geq 0} \left|G(s, \xi)\right|. \tag{3.6.110}$$

It thus follows from (3.6.3) that (with the same R)

$$\liminf_{|\xi| \to\infty} \inf_{t\in\mathbb{R}} \left|G^\infty(t, \xi)\right| > R. \tag{3.6.111}$$

Thus, Corollary 3.6.5 applies with $G^\infty \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and we conclude that $u = 0$. We conclude that $\omega(G)$ is admissible, which is the only remaining hypothesis in Theorem 3.5.2. We apply the theorem, and state the result as an example. For simplicity, we assume that condition (3.6.3) holds for every $R > 0$, as is the case in the examples discussed following the introduction of that condition. Also for simplicity, we do not mention Y^p.

Example 3.6.9. Suppose that $G \colon [0, \infty) \times \mathbb{R} \to \mathbb{R}$ satisfies the three conditions (3.2.22) on page 134 with $I = [0, \infty)$, and also that

$$G(t, \xi)\xi \geq 0, \quad \forall t \geq 0, \forall \xi \in \mathbb{R}. \tag{3.6.112}$$

We also suppose that

$$\liminf_{\xi \to \infty} \inf_{t \geq 0} \bigl|G(t, \xi)\bigr| = \infty. \tag{3.6.113}$$

Then for all

$$f \in L^p\bigl([0, \infty) \times \Omega\bigr) \cap L^\infty\bigl([0, \infty) \times \Omega\bigr) \quad \text{and} \tag{3.6.114}$$

$$g \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \tag{3.6.115}$$

there is a solution $u \in \widehat{\mathscr{W}^p}\bigl([0, \infty)\bigr)$ (see page 114) to the boundary value problem

$$u_t(t, x) - \Delta_x u(t, x) + G\bigl(t, u(t, x)\bigr) = f(t, x), \quad t \geq 0, x \in \Omega; \tag{3.6.116a}$$

$$u(0, x) = g(x), \qquad x \in \Omega; \tag{3.6.116b}$$

$$u(t, x) = 0, \qquad t \geq 0, x \in \partial\Omega, \tag{3.6.116c}$$

$$\lim_{t \to \infty} \sup_{x \in \Omega} \bigl|u(t, x)\bigr| = 0. \tag{3.6.116d}$$

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